

ON THE FAMILY OF PENTAGONAL CURVES OF GENUS 6 AND ASSOCIATED MODULAR FORMS ON THE BALL

KENJI KOIKE

ABSTRACT. In this article we study the inverse of the period map for the family \mathcal{F} of complex algebraic curves of genus 6 equipped with an automorphism of order 5. This is a family with 2 parameters, and is fibred over a certain type of Del Pezzo surface. The period satisfies the hypergeometric differential equation for Appell's $F_1(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{6}{5})$ of two variables after a certain normalization of the variable parameter.

This differential equation and the family \mathcal{F} are studied by G. Shimura (1964), T. Teraada (1983, 1985), P. Deligne - G.D. Mostow (1986) and T. Yamazaki- M. Yoshida (1984). Recently M. Yoshida presented a new approach using the concept of configuration space. Based on their results we show the representation of the inverse of the period map in terms of Riemann theta constants. This is the first variant of the work of H. Shiga (1981) and K. Matsumoto (1989, 2000) to the co-compact case.

0. INTRODUCTION

Let \mathcal{F} be the family of algebraic curves given by

$$C(\lambda) : w^5 = \prod_{i=1}^5 (z - \lambda_i),$$

here the parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ lives on the domain $(\mathbb{P}^1)^5 - \Delta$, where

$$\Delta = \{(\lambda_i) \in (\mathbb{P}^1)^5 : \lambda_i = \lambda_j \text{ for some } i \neq j\}.$$

By putting $(\lambda_1, \lambda_2, \lambda_3) = (0, 1, \infty)$ we can normalize $C(\lambda)$ in the form

$$C'(x, y) : w^5 = z(z-1)(z-x)(z-y)$$

where the parameter (x, y) lives in

$$\Lambda = \{(x, y) \in \mathbb{C}^2 : xy(x-1)(y-1)(x-y) \neq 0\}$$

The period of $C'(x, y)$

$$\eta(x, y) = \int_{\gamma} \frac{dz}{w^2}$$

satisfies the system of differential equation

$$(0.1) \quad \begin{aligned} x(1-x) \frac{\partial^2 u}{\partial x^2} + y(1-x) \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{6}{5} - \frac{11}{5}x\right) \frac{\partial u}{\partial x} - \frac{3}{5}y \frac{\partial u}{\partial y} - \frac{9}{5}u &= 0 \\ y(1-y) \frac{\partial^2 u}{\partial y^2} + x(1-y) \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{6}{5} - 2y\right) \frac{\partial u}{\partial y} - \frac{2}{5}x \frac{\partial u}{\partial x} - \frac{6}{5}u &= 0 \end{aligned}$$

It is the hypergeometric differential equation for the Appell's $F_1(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{6}{5}; x, y)$. The dimension of the solution space is equal to 3. If it holds $\lambda' = g \circ \lambda$ for a certain projective

transformation $g \in \mathrm{PGL}_2(\mathbb{C})$, then we have the biholomorphic equivalence $C(\lambda) \cong C(\lambda')$. So we consider the quotient space

$$X^\circ(2, 5) = ((\mathbb{P}^1)^5 - \Delta) / \mathrm{PGL}_2(\mathbb{C}).$$

as the parameter space for \mathcal{F} , that is biholomorphically equivalent with Λ .

According to the work of T. Terada ([13]), P. Deligne - G.D. Mostow ([1]) and T. Yamazaki - M. Yoshida([15]) we have the following properties :

1. Let $\{\eta_1, \eta_2, \eta_3\}$ be the basis of the solutions of (0.1). The image of the Schwarz map $(x, y) \mapsto [\eta_1(x, y) : \eta_2(x, y) : \eta_3(x, y)] \in \mathbb{P}^2$ is an open dense subset of a 2-dimensional ball \mathbb{B}_2 .
2. The monodromy group for (0.1) is characterized as a certain congruence subgroup of the Picard modular group for $k = \mathbb{Q}(e^{2\pi i/5})$.
3. Let S_5 be the symmetric group of permutations of $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, it has a natural action on $X^\circ(2, 5)$. There is a compactification $X(2, 5)$ of $X^\circ(2, 5)$ so that we have $S_5 \subset \mathrm{Aut}(X(2, 5))$. Yoshida showed $X(2, 5)$ is a Del Pezzo surface of degree 5.
4. We obtain a single valued modular map on \mathbb{B}_2 as the inverse of the Schwarz map.

By the so-called Picard principle we can reduce the period map for \mathcal{F} to the Schwarz map for (0.1). So we proceed our study by the following steps.

In first 4 sections we make up the explicit realization of the above properties (1) - (4):

Section 1. We describe the parameter space $X^\circ(2, 5)$ for \mathcal{F} and its compactification $X(2, 5)$. We list up certain divisors those become to be essential in our study.

Section 2. We construct the period map for \mathcal{F} . And we show how it reduces to the map $\Phi : X^\circ(2, 5) \rightarrow \mathbb{B}_2$.

Section 3. We list up the generator system of the monodromy group for Φ in terms of the unitary reflections.

Section 4. We observe the degeneration of the Schwarz map for (0.1).

Section 5 is the main part of the article. There we study the 0 values of the Riemann theta functions of genus 6 with the characteristic $(a, b) \in (\frac{1}{10}\mathbb{Z})^6 \times (\frac{1}{10}\mathbb{Z})^6$ (Theorem 6.3). These are considered to be a certain kind of automorphic form on \mathbb{B}_2 . Many of the above theta constants identically vanish on \mathbb{B}_2 . At first it is proved that there are only 25 among them those are invariant under the action of the monodromy group. We show they are not identically zero on \mathbb{B}_2^A . Then we determine the vanishing locus on \mathbb{B}_2 of every theta constant in question.

In Section 6 we state the main theorem (Theorem 6.1), that is the representation of Φ^{-1} via the theta constants. As the direct consequence we show the representation of the inverse Schwarz map for the Gauss hypergeometric differential equation $E_{2,1}(\frac{1}{5}, \frac{2}{5}, \frac{4}{5})$. In this case we have the arithmetic triangle group of co-compact type $\Delta(5, 5, 5)$ as the monodromy group, and it is the case mentioned by Shimura [11]. As the another application we show the explicit generator system for the graded ring of the automorphic forms with respect to the unitary group $U(2, 1; \mathcal{O}_k)$ over \mathcal{O}_k (Theorem 6.2).

1. THE CONFIGURATION SPACE $X(2, 5)$

Here we summarize the fundamental facts of $X(2, 5)$. For precise arguments, see [17, Chapter V]. Let $[a : b]$ be a point on \mathbb{P}^1 , and let $\lambda = b/a$ be its representative on $\mathbb{C} \cup \{\infty\}$. Always we use the notation $\lambda_i \in \mathbb{P}^1$ in this sense. Let us consider ordered distinct five points on \mathbb{P}^1 :

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in (\mathbb{P}^1)^5 - \Delta$$

where, Δ is degenerate locus:

$$\Delta = \{(\lambda_i) \in (\mathbb{P}^1)^5 : \lambda_i = \lambda_j \text{ for some } i \neq j\}.$$

A projective transformation $g \in \text{PGL}_2(\mathbb{C})$ acts on $(\mathbb{P}^1)^5$ as

$$g \cdot (\lambda_1, \dots, \lambda_5) = (g(\lambda_1), \dots, g(\lambda_5)).$$

The configuration space $X^\circ(2, 5)$ is defined by the quotient space

$$X^\circ(2, 5) = ((\mathbb{P}^1)^5 - \Delta) / \text{PGL}_2(\mathbb{C}).$$

It has a good compactification

$$X(2, 5) = \overline{X^\circ(2, 5)} = ((\mathbb{P}^1)^5 - \Delta') / \text{PGL}_2(\mathbb{C})$$

where

$$\Delta' = \{(\lambda_i) \in (\mathbb{P}^1)^5 : \lambda_i = \lambda_j = \lambda_k \text{ for some } i \neq j \neq k \neq i\}.$$

There exist ten lines on $X(2, 5)$ of the form

$$L(ij) = \{\text{the orbit of the form } \lambda_i = \lambda_j\} / \text{PGL}_2(\mathbb{C}) \cong \mathbb{P}^1.$$

Notice that $L(ij) \cap L(jk) = \phi$ ($i \neq j \neq k \neq i$) by the definition, and the degenerate locus $X(2, 5) - X^\circ(2, 5)$ is just the union of these ten lines. $X(2, 5)$ is isomorphic to the blow-up of \mathbb{P}^2 at four points. We can see the blow down $\pi : X(2, 5) \rightarrow \mathbb{P}^2$ by the following way. Let us specialize $\lambda_4 = 0$, $\lambda_5 = \infty$ and regard $[\lambda_1 : \lambda_2 : \lambda_3]$ as a point in \mathbb{P}^2 , then we obtain the following correspondence;

$$\begin{aligned} P_1 &= [1 : 0 : 0] = \pi(L(15)), & P_2 &= [0 : 1 : 0] = \pi(L(25)), \\ P_3 &= [0 : 0 : 1] = \pi(L(35)), & P_4 &= [1 : 1 : 1] = \pi(L(45)), \end{aligned}$$

and

$$\pi(X^\circ(2, 5)) = \{[\lambda_1 : \lambda_2 : \lambda_3] \in \mathbb{P}^2 : \lambda_i \neq \lambda_j \text{ } (i \neq j), \quad i, j = 1, 2, 3, 4\}.$$

For five distinct numbers i, j, k, l, m in $\{1, 2, 3, 4, 5\}$, We define a divisor $D(ijklm)$ on $X(2, 5)$ by

$$D(ijklm) = L(ij) + L(jk) + L(kl) + L(lm) + L(mi).$$

Such a divisor is understood as a “juzu sequence” (see [17]). A 5-juzu sequence $(ijklm)$ is the pentagon with vertices i, j, k, l, m in this cyclic order. The divisor $D(ijklm)$ is given by the edges of this pentagon. We identify $(ijklm)$ and $(imlkj)$ since $L(ij) = L(ji)$.

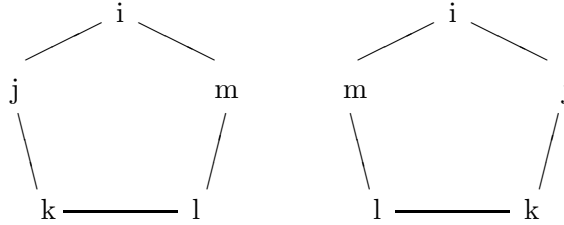


FIGURE 1.

There are twelve different divisors of this form. Let H be a line on \mathbb{P}^2 . As easily shown, $D(ijklm)$ are linearly equivalent to the divisor

$$3\pi^*H - L(15) - L(25) - L(35) - L(45).$$

By the general theory of Del Pezzo surfaces (for example, see [2, Chapter 5]), this is anti-canonical class and very ample. In fact, we have the following proposition by direct calculations.

Proposition 1.1. *Set*

$$J(ijklm)(\lambda) = (\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_l)(\lambda_l - \lambda_m)(\lambda_m - \lambda_i)$$

for twelve $(ijklm)$. Then the map

$$J : X(2, 5) \longrightarrow \mathbb{P}^{11}, \quad J(\lambda) = [\cdots : J(ijklm)(\lambda) : \cdots]$$

is an embedding.

Remark 1.1. *It is necessary to make precise the above notation for $J(ijklm)$. By using the homogeneous coordinate $[a_i : b_i]$ for $\lambda_i \in \mathbb{P}^1$, we set $d(ij) = a_j b_i - a_i b_j$. So $\lambda_i - \lambda_j$ stands for $d(ij)$. The ratio $[J(ijklmn) : J(i'j'k'l'm')]$ defines a rational function on $X(2, 5)$.*

We shall give the correspondence between these divisors and theta functions in a later section.

2. THE FAMILY OF PENTAGONAL CURVES AND THE PERIODS

Let us consider the algebraic curve

$$C_\lambda : y^5 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)(x - \lambda_5),$$

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in (\mathbb{P}^1)^5 - \Delta.$$

If it holds $\lambda' = g \cdot \lambda$ for some $g \in \text{PGL}_2(\mathbb{C})$, then we have a biholomorphic equivalence $C(\lambda) \cong C(\lambda')$. So we identify $C(\lambda)$ and $C(\lambda')$ in this case. Set $\mathcal{F} = \{C_\lambda : \lambda \in X^\circ(2, 5)\}$. We regard C_λ as a five sheeted cyclic covering over \mathbb{P}^1 branched at λ_i via the projection

$$\pi : C_\lambda \longrightarrow \mathbb{P}^1, \quad (x, y) \mapsto x.$$

By the Hurwitz formula, the genus of C_λ is six. We have the following basis of $H^0(C_\lambda, \Omega^1)$:

$$(2.1) \quad \varphi_1 = \frac{dx}{y^2}, \quad \varphi_2 = \frac{dx}{y^3}, \quad \varphi_3 = \frac{xdx}{y^3}, \quad \varphi_4 = \frac{dx}{y^4}, \quad \varphi_5 = \frac{xdx}{y^4}, \quad \varphi_6 = \frac{x^2 dx}{y^4}.$$

Let ρ denotes the automorphism of order five:

$$\rho : C_\lambda \longrightarrow C_\lambda, \quad (x, y) \mapsto (x, \zeta y) \quad (\zeta = \exp(2\pi\sqrt{-1}/5))$$

on C_λ .

Remark 2.1. *Throughout this article always ζ stands for $\exp(2\pi\sqrt{-1}/5)$.*

Next, we construct a symplectic basis of $H_0(C_\lambda, \mathbb{Z})$.

Let $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0) \in X^\circ(2, 5)$ be a real point such that $\lambda_1 < \cdots < \lambda_5$ and C_0 be the corresponding curve. Take a point $x_0 \in \mathbb{P}^1$ such that $\text{Im}(x_0) < 0$, and make line segments l_i connecting x_0 and λ_i . Then $\Sigma = \mathbb{P}^1 - \cup l_i$ is simply connected and $\pi^{-1}(\Sigma)$ is isomorphic to $\Sigma \times \mathbb{Z}/5\mathbb{Z}$. Here, we choose the fiber coordinates $k \in \mathbb{Z}/5\mathbb{Z}$ such that $\rho(x, k) = (x, k + 1)$. Let $\alpha(i, j)$ be the oriented arc from λ_i to λ_j in Σ . We obtain the following five oriented arcs $\alpha_k(i, j)$ ($k = 1, \dots, 5$) in C_0 :

$$(2.2) \quad \alpha_k(i, j) = (\alpha(i, j), k) \subset \Sigma \times \mathbb{Z}/5\mathbb{Z}.$$

We define cycles $\gamma_1, \gamma_2, \gamma_3$ on C_0 (Figure 2) using this notation;

$$(2.3) \quad \begin{aligned} \gamma_1 &= \alpha_1(1, 2) + \alpha_2(2, 1), \\ \gamma_2 &= \alpha_1(3, 4) + \alpha_2(4, 3), \\ \gamma_3 &= \alpha_1(1, 3) + \alpha_2(3, 4) + \alpha_3(4, 2) + \alpha_2(2, 1). \end{aligned}$$

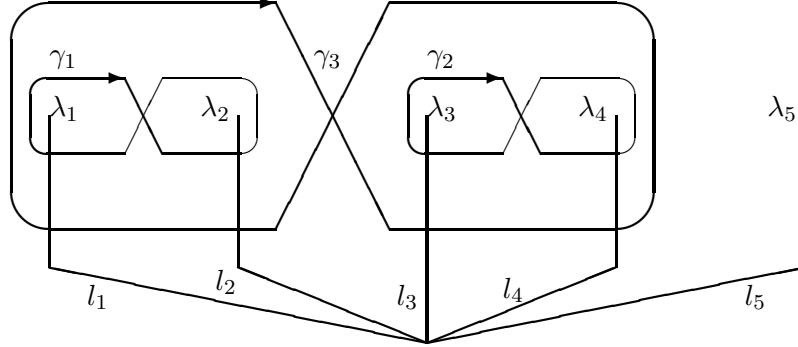


FIGURE 2.

We set

$$(2.4) \quad \begin{aligned} A_1 &= \gamma_1, & A_2 &= \gamma_2, & A_3 &= \gamma_3, & A_4 &= \rho^2(\gamma_1), & A_5 &= \rho^2(\gamma_2), & A_6 &= \rho^4(\gamma_3), \\ B_1 &= \rho(\gamma_1) + \rho^3(\gamma_1), & B_2 &= \rho(\gamma_2) + \rho^3(\gamma_2), & B_3 &= \rho(\gamma_3) + \rho^2(\gamma_3), \\ B_4 &= \rho^3(\gamma_1), & B_5 &= \rho^3(\gamma_2), & B_6 &= \rho(\gamma_3). \end{aligned}$$

The intersection numbers of these cycles are given by

$$A_i \cdot A_j = B_i \cdot B_j = 0, \quad A_i \cdot B_j = \delta_{ij}.$$

So, $\{A_i, B_i\}$ is a symplectic basis of $H_1(C_0, \mathbb{Z})$. Let λ be a point on $X^\circ(2, 5)$, and suppose an arc r from λ^0 to λ . Since the family \mathcal{F} is locally trivial as a topological fiber space over $X^\circ(2, 5)$, by using this trivialization along r , we obtain the systems $\{\alpha_k(i, j)(\lambda)\}$, $\{\gamma_i(\lambda)\}$ and the symplectic basis $\{A_i(\lambda), B_i(\lambda)\}$ on C_λ . We have the relation (2.4) between $\{\gamma_i(\lambda)\}$ and $\{A_i(\lambda), B_i(\lambda)\}$ also. We note that $\{A_i(\lambda), B_i(\lambda)\}$ depend on the homotopy class of r .

Now, we consider the period matrix of C_λ :

$$\Pi(\lambda) = \Pi = (Z_1, Z_2) = \begin{pmatrix} \int_{A_1} \varphi_1 & \cdots & \int_{A_6} \varphi_1 & \int_{B_1} \varphi_1 & \cdots & \int_{B_6} \varphi_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_{A_1} \varphi_6 & \cdots & \int_{A_6} \varphi_6 & \int_{B_1} \varphi_6 & \cdots & \int_{B_6} \varphi_6 \end{pmatrix}.$$

The normalized period matrix $\Omega(\lambda) = \Omega = Z_1^{-1} Z_2$ belongs to the Siegel upper half space of degree 6:

$$\mathfrak{S}_6 = \{\Omega \in \text{GL}_6(\mathbb{C}) : {}^t\Omega = \Omega, \text{Im}(\Omega) \text{ is positive definite}\}.$$

The automorphism ρ acts on $H^0(C_\lambda, \Omega^1)$ and $H_1(C_\lambda, \mathbb{Z})$. So we have the representation matrices $R \in \text{GL}_6(\mathbb{C})$ and $M \in \text{GL}_{12}(\mathbb{Z})$ of ρ with respect to the basis $\{\varphi_i\}$ and $\{A_i, B_i\}$, respectively. It holds $R\Pi = \Pi M$. Put

$$(2.5) \quad M = \begin{pmatrix} {}^tD & {}^tB \\ {}^tC & {}^tA \end{pmatrix}, \quad \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then the matrix σ belongs to the symplectic group

$$\mathrm{Sp}_{12}(\mathbb{Z}) = \{g \in \mathrm{GL}_{12}(\mathbb{Z}) : {}^t g J g = J\}, \quad J = \begin{pmatrix} 0 & I_6 \\ -I_6 & 0 \end{pmatrix}$$

and it holds

$$\Omega = (A\Omega + B)(C\Omega + D)^{-1}.$$

As easily shown, φ_i ($i = 1, \dots, 6$) is eigenvectors of ρ and we have

$$R = \begin{pmatrix} \zeta^3 & & & & & \\ & \zeta^2 & & & & \\ & & \zeta^2 & & & \\ & & & \zeta & & \\ & 0 & & & \zeta & \\ & & & & & \zeta \end{pmatrix}.$$

By the relation (2.4) of A_i, B_i , we have

$$(2.6) \quad \Pi = (a, b, c, R^2 a, R^2 b, R^4 c, (R + R^3)a, (R + R^3)b, (R + R^2)c, R^3 a, R^3 b, Rc),$$

where we denote

$$a = {}^t(\int_{\gamma_1} \varphi_1, \dots, \int_{\gamma_1} \varphi_6), \quad b = {}^t(\int_{\gamma_2} \varphi_1, \dots, \int_{\gamma_2} \varphi_6), \quad c = {}^t(\int_{\gamma_3} \varphi_1, \dots, \int_{\gamma_3} \varphi_6).$$

According to (2.4),

$$\rho(A_1) = \rho(\gamma_1) = (\rho(\gamma_1) + \rho^3(\gamma_1)) - \rho^3(\gamma_1) = B_1 - B_4.$$

By the same way, we can describe $\rho(A_2), \dots, \rho(B_6)$ in terms of $\{A_i, B_i\}$. So we can determine M , and obtain

$$(2.7) \quad \sigma = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Put

$$\eta(\lambda) = [\eta_1(\lambda) : \eta_2(\lambda) : \eta_3(\lambda)] \in \mathbb{P}^2, \quad \eta_1(\lambda) = \int_{\gamma_1} \varphi_1, \quad \eta_2(\lambda) = \int_{\gamma_2} \varphi_1, \quad \eta_3(\lambda) = \int_{\gamma_3} \varphi_1.$$

These are multi-valued analytic functions of λ . Applying the Riemann positive condition

$$(\int_{A_1} \varphi_1, \dots, \int_{B_6} \varphi_1) J {}^t(\int_{A_1} \overline{\varphi}_1, \dots, \int_{B_6} \overline{\varphi}_1) > 0$$

for (2.6), we obtain

$$|\eta_1|^2 + |\eta_2|^2 + \frac{1 - \sqrt{5}}{2} |\eta_3|^2 < 0.$$

So, $\eta = (\eta_1, \eta_2, \eta_3)$ belongs to the complex ball

$$(2.8) \quad \mathbb{B}_2^A = \{\eta \in \mathbb{P}^2 : {}^t\bar{\eta}A\eta < 0\}, \quad A = \text{diag}(1, 1, \frac{1 - \sqrt{5}}{2}).$$

Next, we determine Ω explicitly. Write $a = (a_i)$, $b = (b_i)$ and $c = (c_i)$. Then, the Riemann bilinear relation $\Pi J^t \Pi = 0$ induces the following equations:

$$c_2 = -(\zeta^2 + \zeta^3)(a_1 a_2 + b_1 b_2)/c_1, \quad c_3 = -(\zeta^2 + \zeta^3)(a_1 a_3 + b_1 b_3)/c_1.$$

By substituting them for Z_1, Z_2 in Π we can proceed the calculation of $\Omega = Z_1^{-1} Z_2$ (using a computer). Hence we have the following:

Lemma 2.1. *Let $\Delta = \eta_1^2 + \eta_2^2 - \zeta^3(1 + \zeta)\eta_3^2$. The period matrix $\Omega = (\Omega_{ij})$ is given by*

$$\begin{aligned} \Omega_{11} &= (\zeta^3 - 1)(\eta_1^2 + (1 + \zeta^3)\eta_2^2 + \eta_3^2)/\Delta, & \Omega_{44} &= -\zeta^2(\eta_1^2 + \zeta^2\eta_2^2 - (1 + \zeta)\eta_3^2)/\Delta, \\ \Omega_{22} &= (\zeta^3 - 1)((1 + \zeta^3)\eta_1^2 + \eta_2^2 + \eta_3^2)/\Delta, & \Omega_{55} &= -\zeta^2(\zeta^2\eta_1^2 + \eta_2^2 - (1 + \zeta)\eta_3^2)/\Delta, \\ \Omega_{33} &= (\zeta^2 - 1)(\eta_1^2 + \eta_2^2 - \zeta^3\eta_3^2)/\Delta, & \Omega_{66} &= -\zeta^3(\eta_1^2 + \eta_2^2 - (1 + \zeta^4)\eta_3^2)/\Delta, \\ \Omega_{12} &= (\zeta^3 - \zeta)\eta_1\eta_2/\Delta, & \Omega_{45} &= (\zeta^4 - \zeta^2)\eta_1\eta_2/\Delta, \\ \Omega_{15} &= (\zeta^4 - \zeta)\eta_1\eta_2/\Delta, & \Omega_{24} &= (\zeta^4 - \zeta)\eta_1\eta_2/\Delta, \\ \Omega_{13} &= (1 - \zeta^2)\eta_1\eta_3/\Delta, & \Omega_{23} &= (1 - \zeta^2)\eta_2\eta_3/\Delta, \\ \Omega_{46} &= (\zeta^4 - \zeta)\eta_1\eta_3/\Delta, & \Omega_{56} &= (\zeta^4 - \zeta)\eta_2\eta_3/\Delta, \\ \Omega_{16} &= (\zeta^3 - \zeta)\eta_1\eta_3/\Delta, & \Omega_{26} &= (\zeta^3 - \zeta)\eta_2\eta_3/\Delta, \\ \Omega_{34} &= (1 - \zeta^3)\eta_1\eta_3/\Delta, & \Omega_{35} &= (1 - \zeta^3)\eta_2\eta_3/\Delta, \\ \Omega_{14} &= \zeta^3((1 + \zeta)\eta_1^2 + (1 + \zeta^3)\eta_2^2 + \eta_3^2)/\Delta, & \Omega_{25} &= \zeta^3((1 + \zeta^3)\eta_1^2 + (1 + \zeta)\eta_2^2 + \eta_3^2)/\Delta, \\ \Omega_{36} &= (\zeta + \zeta^2)(\eta_1^2 + \eta_2^2 - \zeta^3(1 + \zeta^2)\eta_3^2)/\Delta. \end{aligned}$$

Now we define our period map

$$\Phi : X^\circ(2, 5) \longrightarrow \mathbb{B}_2^A, \quad \lambda \mapsto [\eta_1(\lambda) : \eta_2(\lambda) : \eta_3(\lambda)],$$

that is multi-valued analytic. The above Lemma says that the original period map $\lambda \mapsto \Omega(\lambda)$ factors as

$$X^\circ(2, 5) \longrightarrow \mathbb{B}_2^A \longrightarrow \mathfrak{S}_6.$$

Throughout this paper, we denote matrices of the form in Lemma 2.1 by $\Omega(\eta)$.

3. THE MONODROMY GROUP AND REFLECTIONS

The multi-valuedness of Φ induces a unitary representation with respect to A in (2.8) of the fundamental group $\pi_1(X^\circ(2, 5))$. We call it the monodromy group of Φ . The structures of our monodromy group is studied in [15]. Set

$$\Gamma = \{g \in \text{GL}_3(\mathbb{Z}[\zeta]) : {}^t\bar{g}Ag = A\}, \quad \Gamma(1 - \zeta) = \{g \in \Gamma : g \equiv I_3 \pmod{1 - \zeta}\}.$$

The group Γ acts on \mathbb{B}_2^A (left action).

Theorem 3.1 (T. Yamazaki, M. Yoshida [15]). (1). *The monodromy group of the period map Φ coincides with $\Gamma(1 - \zeta)$ and the quotient $\Gamma/(\pm I)\Gamma(1 - \zeta)$ is isomorphic to the symmetric group S_5 .*

(2). *The quotient $\mathbb{B}_2^A/\Gamma(1 - \zeta)$ is biholomorphically equivalent to the blow up of \mathbb{P}^2 at four points.*

Remark 3.1 (see [15]). *There are ten (-1) -curves on $\mathbb{B}_2^A/\Gamma(1-\zeta)$, and S_5 acts transitively on them.*

According to [13] and [15], it is proved that Γ and $\Gamma(1-\zeta)$ are reflection groups and the generator systems are given also. We expose those generator system in a form adapted for our calculation in the later sections.

Let us consider the reference point $\lambda^0 \in X^\circ(2, 5)$ again. Now we define the half way monodromy transformation g_{12} induced from the permutation of λ_1^0 and λ_2^0 . Let us consider a continuous arc R_{12} starting from λ^0 :

$$(3.1) \quad \lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3^0, \lambda_4^0, \lambda_5^0), \quad (0 \leq t \leq 1)$$

such that (Figure 3)

$$\lambda_2(1) = \lambda_1^0, \quad \lambda_1(1) = \lambda_2^0, \quad \text{Im}(\lambda_1(t)) < 0 < \text{Im}(\lambda_2(t)) \quad (0 < t < 1).$$

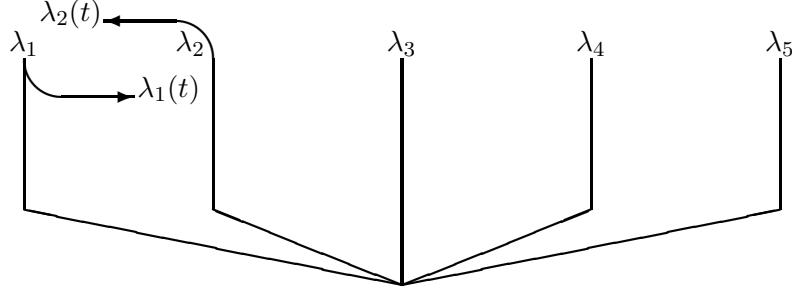


FIGURE 3.

Let $\eta(t) = \eta(\lambda(t))$ be the corresponding periods. Recall the definition (2.3). It is apparent that γ_2 and γ_3 are invariant after this deformation process. Describing $\gamma_1(t)$ for any $0 \leq t \leq 1$, we get $\gamma_1(1) = -\rho(\gamma_1(0))$. Namely,

$$\begin{pmatrix} \eta_1(1) \\ \eta_2(1) \\ \eta_3(1) \end{pmatrix} = \begin{pmatrix} -\zeta^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1(0) \\ \eta_2(0) \\ \eta_3(0) \end{pmatrix}.$$

The matrix in the right hand side belongs to Γ , and we denote it by g_{12} . We define g_{23} , g_{34} , g_{45} by the same manner. Set

$$(3.2) \quad h_{12} = (g_{12})^2, \quad h_{23} = (g_{23})^2, \quad h_{34} = (g_{34})^2$$

$$(3.3) \quad h_{13} = (g_{23})^{-1}(g_{12})^2 g_{23}, \quad h_{14} = (g_{23}g_{34})^{-1}(g_{12})^2 g_{23}g_{34}.$$

Proposition 3.1 (see [16]). *The monodromy group is generated by h_{ij} in (3.2).*

Let T_α be the reflection on \mathbb{B}_2^A with respect to a root α ;

$$T_\alpha(\eta) = \eta - (1 + \zeta^3) \frac{{}^t \bar{\alpha} A \eta}{{}^t \bar{\alpha} A \alpha} \alpha,$$

and R_β be the reflection on \mathbb{B}_2^A with respect to a root β ;

$$R_\beta(\eta) = \eta - (1 - \zeta) \frac{{}^t \bar{\beta} A \eta}{{}^t \bar{\beta} A \beta} \beta.$$

Lemma 3.1. *Set*

and set

[illegible]

By same consideration, we obtain following;

$$(3.5) \quad \hat{g}_{23} = \begin{pmatrix} 1 & 1 & 0 & 1 & -1 & 1 & 2 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 0 & -2 & 2 & 0 & -1 & 1 & -2 \\ 1 & -1 & 2 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & 1 & 2 & -2 & 0 & 1 & -1 & 2 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & -2 & -1 & 1 & 1 & -1 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 2 & 0 & -1 & -2 & 0 & -1 & 1 & -1 & 0 & 2 \end{pmatrix},$$

$$(3.6) \quad \hat{g}_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(3.7) \quad \hat{g}_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. DEGENERATE LOCI

According to Theorem 3.1, P. Deligne - G.D. Mostow([1]) and T. Terada([13]) the period map Φ induces the biholomorphic equivalence

$$\tilde{\Phi} : X^\circ(2, 5) \xrightarrow{\sim} \mathbb{B}_2^\circ / \Gamma(1 - \zeta),$$

where $\mathbb{B}_2^\circ = \text{Im}\Phi$. Moreover we have the unique extension

$$\tilde{\Phi} : X(2, 5) \xrightarrow{\sim} \mathbb{B}_2^A / \Gamma(1 - \zeta),$$

and $\cup L(ij) = X(2, 5) - X^\circ(2, 5)$ corresponds to $(\mathbb{B}_2^A - \mathbb{B}_2^\circ)/\Gamma(1 - \zeta)$. Let π be the projection $\mathbb{B}_2^A \rightarrow \mathbb{B}_2^A/\Gamma(1 - \zeta)$, and let $\ell(ij)$ denote $\pi^{-1}(\tilde{\Phi}(L(ij)))$.

Now we consider a degenerate curve

$$y^5 = (x - \lambda_1)^2(x - \lambda_3)(x - \lambda_4)(x - \lambda_5)$$

with $(\lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_5) \in L(12)$, and putting $\lambda' = (\lambda_1, \lambda_3, \lambda_4, \lambda_5)$ we denote it by $C_{\lambda'}$. Let $\tilde{C}_{\lambda'}$ denote the non-singular model of $C_{\lambda'}$. It is a curve of genus 4. Set \mathcal{F}_{12} be the totality of $\tilde{C}_{\lambda'}$. For the parameter $(\lambda^0)' = (\lambda_1^0, \lambda_3^0, \lambda_4^0, \lambda_5^0)$ the cycle γ_1 vanishes on $\tilde{C}_{(\lambda^0)'}$, but γ_2 and γ_3 are still alive. So we can define A_i, B_i ($i = 2, 3, 5, 6$) on $\tilde{C}_{\lambda'}$ by the same argument as for C_λ . Hence we obtain a basis $\{A_i, B_i\}$ ($i = 2, 3, 5, 6$) of $H_1(\tilde{C}_{\lambda'}, \mathbb{Z})$. By putting $\lambda' = (0, 1, t, \infty)$ the period

$$(4.1) \quad \int_{\gamma} x^{-\frac{4}{5}}(x-1)^{-\frac{2}{5}}(x-t)^{-\frac{2}{5}}dx, \quad (\gamma \in H_1(\tilde{C}_{\lambda'}, \mathbb{Z}))$$

on $\tilde{C}_{\lambda'}$ gives a solution for the Gauss hypergeometric differential equation $E_{2,1}(\frac{1}{5}, \frac{2}{5}, \frac{4}{5})$:

$$(4.2) \quad t(1-t)\frac{d^2u}{dt^2} + \left(\frac{4}{5} - \frac{8}{5}t\right)\frac{du}{dt} - \frac{2}{5}u = 0$$

The corresponding monodromy group is the triangle group $\Delta(5, 5, 5)$ (see [12], [16], [17, p.138]). Set

$$\mathbb{B}_1 = \{\eta \in \mathbb{B}_2^A : \eta_1 = 0\},$$

it is the mirror of the reflection g_{12} . By using the system $\{\gamma_2, \gamma_3\}$ we define a multi-valued map

$$\Phi_{12} : L(12) \longrightarrow \mathbb{B}_1, \quad \lambda \mapsto [0 : \eta_2(\lambda) : \eta_3(\lambda)].$$

It induces the restriction $\tilde{\Phi}|_{L(12)}$. By the same manner we obtain that $\tilde{\Phi}|_{L(ij)}$ is the mirror of the reflection g_{ij} . Suppose $\lambda \in L(12)$ and set $\eta = \eta(\lambda) = \Phi_{12}(\lambda)$. By putting $\eta_1 = 0$ in Lemma 2.1, we see that

$$(4.3) \quad \Omega(\eta) = \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} \oplus \Omega'(\eta), \quad \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} = \tau_0 = \begin{pmatrix} \zeta - 1 & \zeta^2 + \zeta^3 \\ \zeta^2 + \zeta^3 & -\zeta^4 \end{pmatrix}$$

with a certain element $\Omega'(\eta) \in \mathfrak{S}_4$. Moreover, in case $\eta_0 = [0 : 0 : 1] \in \ell(12) \cap \ell(34)$ we have

$$(4.4) \quad \Omega(\eta_0) = \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} \oplus \begin{pmatrix} \Omega_{22} & \Omega_{25} \\ \Omega_{52} & \Omega_{55} \end{pmatrix} \oplus \begin{pmatrix} \Omega_{33} & \Omega_{36} \\ \Omega_{63} & \Omega_{66} \end{pmatrix} = \tau_0 \oplus \tau_0 \oplus \tau_0$$

We use the above matrix to numerical evaluation of theta functions in later section.

5. THETA FUNCTIONS

5.1. Invariant theta characteristics. We recall basic facts on the Riemann theta functions. For a characteristic $(a, b) \in (\mathbb{R}^g)^2$, the theta function $\Theta_{(a,b)}(z, \Omega)$ on $\mathbb{C}^g \times \mathfrak{S}_g$ is defined by the series

$$\Theta_{(a,b)}(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp[\pi \sqrt{-1}^t(n+a)\Omega(n+a) + 2\pi \sqrt{-1}^t(n+a)(z+b)].$$

These functions satisfy the following period relations

$$(5.1) \quad \Theta_{(a,b)}(z+m, \Omega) = \exp(2\pi \sqrt{-1}^t m a) \Theta_{(a,b)}(z, \Omega),$$

$$(5.2) \quad \Theta_{(a,b)}(z + \Omega m, \Omega) = \exp(-\pi\sqrt{-1}^t m \Omega m - 2\pi\sqrt{-1}^t m(z + b)) \Theta_{(a,b)}(z, \Omega)$$

for $m \in \mathbb{Z}^g$. For the characteristics, we have

$$(5.3) \quad \Theta_{(a+n, b+m)}(z, \Omega) = \exp(2\pi\sqrt{-1}^t a m) \Theta_{(a,b)}(z, \Omega),$$

for $n, m \in \mathbb{Z}^g$ and

$$(5.4) \quad \Theta_{(-a, -b)}(z, \Omega) = \Theta_{(a,b)}(z, \Omega).$$

Theta constants $\Theta_{(a,b)}(\Omega) = \Theta_{(a,b)}(0, \Omega)$ satisfy following transformation formula (see [4, p176]) as function on \mathfrak{S}_g . For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SP}_{2g}(\mathbb{Z})$, set

$$(5.5) \quad g\Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

$$(5.6) \quad g(a, b) = (Da - Cb, -Ba + Ab) + \frac{1}{2}((C^t D)_0, (A^t B)_0)$$

$$(5.7) \quad \phi_{(a,b)}(g) = -\frac{1}{2}({}^t a^t D B a - 2{}^t a^t B C b + {}^t b^t C A b) + \frac{1}{2}({}^t a^t D - {}^t b^t C)(A^t B)_0$$

where $(A)_0$ stands for the diagonal vector of a matrix A . Then we have

$$(5.8) \quad \Theta_{g(a,b)}(g\Omega) = \kappa(g) \exp(2\pi\sqrt{-1}\phi_{a,b}(g)) \det(C\Omega + D)^{\frac{1}{2}} \Theta_{(a,b)}(\Omega)$$

where, $\kappa(g)$ is a certain 8-th root of 1 depending only on g .

Remark 5.1. By the definition, we have

$$\Theta_{(a,b)}(z, \Omega) = \exp(\pi\sqrt{-1}^t a \Omega a + 2\pi\sqrt{-1}^t a(z + b)) \Theta_{(0,0)}(z + \Omega a + b, \Omega),$$

so we often identify a characteristic $(a, b) \in (\mathbb{R}^g)^2$ with $\Omega a + b \in \mathbb{C}^g$. For $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, we have

$$\Omega'(Da - Cb) + (-Ba + Ab) = {}^t(C\Omega + D)^{-1}(\Omega a + b),$$

where $\Omega' = (A\Omega + B)(C\Omega + D)^{-1}$.

For these formulas, see [4] and [8].

Henceforth we suppose the characteristics (a, b) satisfy $a, b \in (\frac{1}{10}\mathbb{Z})^6$.

Lemma 5.1. Let σ be the matrix in (2.5) and write $a = (a_i)$, $b = (b_i)$.

1. We have

$$\sigma(a, b) \equiv (a, b) \pmod{\mathbb{Z}}$$

if and only if

$$\begin{aligned} 5a_1 &\equiv \frac{1}{2}, & a_4 &\equiv a_1, & b_1 &\equiv -2a_1, & b_4 &\equiv -a_1 \\ 5a_2 &\equiv \frac{1}{2}, & a_5 &\equiv a_2, & b_2 &\equiv -2a_2, & b_5 &\equiv -a_2 \\ 5a_3 &\equiv \frac{1}{2}, & a_6 &\equiv a_3, & b_3 &\equiv -2a_3, & b_6 &\equiv -a_3 \end{aligned} \pmod{\mathbb{Z}}.$$

(2) Let (a, b) be the characteristic with the above condition. Then we have

$$\hat{g}(a, b) \equiv (a, b) \pmod{\mathbb{Z}} \quad \text{for all } g \in \Gamma(1 - \zeta).$$

Proof. (1) Using the exact form (2.7) we can describe $\sigma(a, b)$. Then we deduce the assertion.

(2) The transformation $g(a, b)$ in (5.6) define a group action of the symplectic group on $(\mathbb{R}/\mathbb{Z})^{2g}$ (see [4]). We can check that the equality for every member of the generator system $\{h_{ij}\}$ of $\Gamma(1 - \zeta)$. \square

Definition 5.1. Let (a, b) be the characteristic satisfying the condition Lemma 5.1 (1). Then we can put

$$(5.9) \quad a = \frac{1}{10} {}^t(a_1, a_2, a_3, a_1, a_2, a_3), \quad b = \frac{1}{10} {}^t(-2a_1, -2a_2, -2a_3, -a_1, -a_2, -a_3).$$

Let (a_1, a_2, a_3) denote this characteristic. We call $(a, b) = (a_1, a_2, a_3)$ “ σ -invariant” if a_1, a_2, a_3 are odd integers. For a characteristic of this type, we denote the zero locus of $\Theta_{(a_1, a_2, a_3)}$ on \mathbb{B}_2^A by $\vartheta(a_1, a_2, a_3)$;

$$\vartheta(a_1, a_2, a_3) = \{\eta \in \mathbb{B}_2^A : \Theta_{(a_1, a_2, a_3)}(\Omega(\eta)) = 0\}.$$

Remark 5.2. By the transformation formula (5.8) and Lemma 5.1, we see that

$$\Theta_{(a_1, a_2, a_3)}(g\Omega(\eta)) = (a \text{ unit function}) \times \Theta_{(a_1, a_2, a_3)}(\Omega(\eta))$$

for a invariant characteristic (a_1, a_2, a_3) and $g \in \Gamma(1 - \zeta)$. Hence if we have $\eta \in \vartheta(a_1, a_2, a_3)$, then $\Gamma(1 - \zeta)$ -orbit of η contained in $\vartheta(a_1, a_2, a_3)$.

Lemma 5.2. Let (a_1, a_2, a_3) be a σ -invariant characteristic. If $2a_1^2 + 2a_2^2 + a_3^2 \notin 5\mathbb{Z}$, then $\vartheta(a_1, a_2, a_3) = \mathbb{B}_2^A$. Namely, $\Theta_{(a_1, a_2, a_3)}$ vanishes on \mathbb{B}_2^A .

Proof. We apply the transformation formula (5.8) for $g = \sigma^4$.

For it we proceed the preparatory calculations. At first, get the explicit form of $g = \sigma^4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ by using (2.7). So we obtain

$$\phi_{(a_1, a_2, a_3)}(\sigma^4) = \frac{1}{40}(2a_1^2 + 2a_2^2 + a_3^2).$$

Using the explicit form of $\Omega(\eta)$ in Lemma 2.1, we get

$$\det(C\Omega(\eta) + D) = 1$$

for all $\eta \in \mathbb{B}_2^A$ by a computer and calculation. By (5.3), we may put

$$\Theta_{\sigma^4(a_1, a_2, a_3)}(\Omega) = \exp[2\pi\sqrt{-1}^t am] \Theta_{(a_1, a_2, a_3)}(\Omega)$$

for a certain $m \in \mathbb{Z}^6$. Returning to the explicit form of $\sigma^4(a_1, a_2, a_3)$ we should get m . We check that $\exp[2\pi\sqrt{-1}^t am] = 1$ by a computer aided calculation. Hence we have

$$\Theta_{(a_1, a_2, a_3)}(\Omega(\eta)) = \kappa(\sigma^4) \exp\left[\frac{1}{20}\pi\sqrt{-1}(2a_1^2 + 2a_2^2 + a_3^2)\right] \Theta_{(a_1, a_2, a_3)}(\Omega(\eta))$$

for all $\eta \in \mathbb{B}_2^A$. This implies our assertion since $\kappa(\sigma^4)$ is an 8-th root of 1. \square

We consider odd integers a_1, a_2, a_3 modulo $10\mathbb{Z}$. There exist 25 representatives of the σ -invariant characteristic (a_1, a_2, a_3) satisfying the condition $2a_1^2 + 2a_2^2 + a_3^2 \in 5\mathbb{Z}$;

$$(5.10) \quad \begin{aligned} & (1, 1, 1), (1, 1, 9), (1, 9, 1), (9, 1, 1), (1, 3, 5), (1, 7, 5), \\ & (3, 1, 5), (7, 1, 5), (3, 3, 3), (3, 3, 7), (3, 7, 3), (7, 3, 3), \end{aligned}$$

and

$$(5.11) \quad \begin{aligned} & (9, 9, 9), (9, 9, 1), (9, 1, 9), (1, 9, 9), (9, 7, 5), (9, 3, 5), \\ & (7, 9, 5), (3, 9, 5), (7, 7, 7), (7, 7, 3), (7, 3, 7), (3, 7, 7), \end{aligned}$$

and $(5, 5, 5)$.

Remark 5.3. (1) *The characteristic $(5, 5, 5)$ is an odd half integer characteristic (see [8]), hence $\Theta_{(5,5,5)}(\Omega)$ vanishes identically.*

(2) *By (5.3) and (5.4), we see that $\Theta_{(a_1, a_2, a_3)}(\Omega)$ is a scalar multiple of $\Theta_{(b_1, b_2, b_3)}(\Omega)$ if $a_1 + b_1, a_2 + b_2, a_3 + b_3 \in 10\mathbb{Z}$. So the system in (5.10) and the system in (5.11) are essentially the same.*

Lemma 5.3. *Let (a_1, a_2, a_3) be a member of the system (5.10)(equivalently (5.11)). The group Γ acts on the set of twelve $\vartheta(a_1, a_2, a_3)$ transitively.*

Proof. We have an explicit form of \hat{g}_{ij} in (3.4) – (3.7). We use it and obtain

$$\hat{g}_{12}(a_1, a_2, a_3) \equiv (-a_1, a_2, a_3), \quad \hat{g}_{34}(a_1, a_2, a_3) \equiv (a_1, -a_2, a_3),$$

(a_1, a_2, a_3)	$\hat{g}_{23}(a_1, a_2, a_3) \equiv$	$\hat{g}_{45}(a_1, a_2, a_3) \equiv$
(1,1,1)	(3,3,7)	(1,9,9)
(1,1,9)	(7,7,7)	(1,7,5)
(1,9,1)	(9,7,5)	(1,3,5)
(9,1,1)	(7,9,5)	(9,9,9)
(1,3,5)	(9,1,9)	(1,9,1)
(1,7,5)	(7,3,3)	(1,1,9)
(3,1,5)	(1,9,9)	(3,3,7)
(7,1,5)	(3,7,3)	(7,3,7)
(3,3,3)	(9,9,1)	(3,7,7)
(3,3,7)	(1,1,1)	(3,1,5)
(3,7,3)	(7,1,5)	(3,9,5)
(7,3,3)	(1,7,5)	(7,7,7)

According to (5.8),

$$g(\vartheta(a_1, a_2, a_3)) = \vartheta(\hat{g}(a_1, a_2, a_3))$$

So the assertion follows. \square

5.2. The zero loci of twelve theta functions. Here we state Riemann's theorem. Let C be an algebraic curve of genus g , let $\{A_i, B_i\}$ be a symplectic basis of $H_1(C, \mathbb{Z})$ such that $A_i \cdot B_j = \delta_{ij}$, and let $\{\omega_i\}$ be the basis of $H^0(C, \Omega^1)$ such that $\int_{A_i} \omega_j = \delta_{ij}$. Then $\Omega = (\int_{B_i} \omega_j)$ belongs to \mathfrak{S}_g . We denote ${}^t(\int_{\gamma} \omega_1, \dots, \int_{\gamma} \omega_g)$ by $\int_{\gamma} \omega$.

Theorem 5.1 (see [8], p149). *Let us fix a point $P_0 \in C$. Then there is a vector $\Delta \in \mathbb{C}^g$, such that for all $z \in \mathbb{C}^g$, multi-valued function*

$$f(P) = \Theta_{(0,0)}(z + \int_{P_0}^P \omega, \Omega) \quad (P \in C)$$

on C either vanishes identically, or has g zeros Q_1, \dots, Q_g with

$$\sum_{i=1}^g \int_{P_0}^{Q_i} \omega \equiv -z + \Delta \mod \Omega\mathbb{Z}^g + \mathbb{Z}^g.$$

Remark 5.4 (see [8]). (1) The vector Δ in the theorem is called the Riemann constant, and depends on the symplectic basis $\{A_i, b_i\}$ and the base point P_0 . For the fixed $\{A_i, B_i\}$ and P_0 , Δ is uniquely determined as the point of the Jacobian $J(C) = \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)$ by the property of the theorem.

(2) If we take P_0 such that the divisor $(2g-2)P_0$ is linearly equivalent to the canonical divisor, then we have $\Delta \in \frac{1}{2}\Omega\mathbb{Z} + \frac{1}{2}\mathbb{Z}$.

Corollary 5.1 (see [8]). Under same situation as the theorem, $\Theta_{a,b}(\Omega) = 0$ if and only if there exist $Q_1, \dots, Q_g \in C$ such that

$$\Delta - (\Omega a + b) \equiv \sum_{i=1}^{g-1} \int_{P_0}^{Q_i} \omega.$$

Now, let us return to our case. Let $\lambda^0 \in X^\circ(2, 5)$ and C_0 be as in section 2 and $\omega_1, \dots, \omega_6$ be the basis of $H^0(C_0, \Omega^1)$ such that $\int_{A_i} \omega_j = \delta_{ij}$. We denote the ramified points over $\lambda_i \in \mathbb{P}^1$ by $P_i \in C_0$. Let us take the base point P_0 arbitrary among $\{P_1, \dots, P_5\}$ and Δ_0 be the Riemann constant with respect to $\{A_i, B_i\}$ and P_0 .

Lemma 5.4. The Riemann constant Δ_0 corresponds to the characteristic $(5, 5, 5)$.

Proof. The divisor of the holomorphic 1-form $(x - \lambda_i)^2 dx/y^4$ is $10P_i$. Hence Δ_0 is a half integer characteristic (see Remark 5.4). For $z = \Omega a + b$ ($a, b \in \mathbb{R}^6$) and $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, applying (5.8) we have

$$(5.12) \quad \Theta_{\sigma(\Delta_0 - z)}(\Omega) = (\text{a unit function}) \times \Theta_{\Delta_0 - z}(\Omega)$$

since $\sigma\Omega = \Omega$. By (5.6) and Remark 5.1, we have

$$\sigma(\Delta_0 - z) = \sigma\Delta_0 - {}^t(C\Omega + D)^{-1}z.$$

Hence it holds

$$\begin{aligned} \Theta_{\sigma\Delta_0 - {}^t(C\Omega + D)^{-1}z}(\Omega) = 0 &\Leftrightarrow \Theta_{\Delta_0 - z}(\Omega) = 0 \\ &\Leftrightarrow z \equiv \sum_{i=1}^5 \int_{P_0}^{Q_i} \omega \quad \text{for } \exists Q_1, \dots, Q_5 \in C_0 \end{aligned}$$

by Corollary 5.1. Namely, putting $w = {}^t(C\Omega + D)^{-1}z$ we have

$$\Theta_{\sigma\Delta_0 - w}(\Omega) = 0 \Leftrightarrow {}^t(C\Omega + D)w \equiv \sum_{i=1}^5 \int_{P_0}^{Q_i} \omega \quad \text{for } \exists Q_1, \dots, Q_5 \in C_0.$$

Let us recall that σ is the symplectic representation matrix of ρ with respect to the basis $\{A_i, B_i\}$ of $H_1(C_0, \mathbb{Z})$. And we have

$$\begin{pmatrix} I & \Omega \end{pmatrix} \begin{pmatrix} {}^tD & {}^tB \\ {}^tC & {}^tA \end{pmatrix} = ({}^t(C\Omega + D) \quad {}^t(A\Omega + B)) = {}^t(C\Omega + D) \begin{pmatrix} I & \Omega \end{pmatrix},$$

so ${}^t(C\Omega + D)$ is the representation matrix of ρ with respect to the basis $\{\omega_1, \dots, \omega_6\}$ of $H^0(C_0, \Omega^1)$. Hence it holds

$$\Theta_{\sigma\Delta_0 - w}(\Omega) = 0 \Leftrightarrow w \equiv \sum_{i=1}^5 \int_{P_0}^{Q_i} (\rho^{-1})^* \omega \equiv \sum_{i=1}^5 \int_{\rho^{-1}(P_0)}^{\rho^{-1}(Q_i)} \omega \equiv \sum_{i=1}^5 \int_{P_0}^{\rho^{-1}(Q_i)} \omega$$

Recalling Remark 5.4 (1), this implies that $\sigma\Delta_0$ is the Riemann constant, that is $\sigma\Delta_0 \equiv \Delta_0$. Hence we have $\Delta_0 \equiv (5, 5, 5)$ since $(5, 5, 5)$ is the unique σ -invariant half integer characteristic. \square

Next, let us consider the oriented arcs $\alpha_k(i, j)$ defined by (2.2) and the integrals $\int_{\alpha_k(i, j)} \omega \in \mathbb{C}^6$.

Lemma 5.5. *The integral $\int_{\alpha_k(i, j)} \omega$ is a five torsion point $\Omega a + b$ on $\mathbb{C}^6/(\Omega\mathbb{Z}^6 + \mathbb{Z}^6)$ of the form*

$$a = \frac{1}{10} {}^t(a_1, a_2, a_3, a_1, a_2, a_3), \quad b = \frac{1}{10} {}^t(-2a_1, -2a_2, -2a_3, -a_1, -a_2, -a_3)$$

with $a_1, a_2, a_3 \in 2\mathbb{Z}$. In explicit way, it holds

$$\int_{\alpha_k(1,2)} \omega \equiv (6, 0, 0), \quad \int_{\alpha_k(1,3)} \omega \equiv (8, 2, 6), \quad \int_{\alpha_k(1,4)} \omega \equiv (8, 8, 6), \quad \int_{\alpha_k(1,5)} \omega \equiv (8, 0, 8) \pmod{\Omega\mathbb{Z}^6 + \mathbb{Z}^6}$$

with the same notation in Definition 5.1 and identification referred in Remark 5.1 (Note that any $\alpha_k(i, j)$ is written as a combination of $\alpha_k(1, 2)$, $\alpha_k(1, 3)$, $\alpha_k(1, 4)$ and $\alpha_k(1, 5)$).

Proof. Since $D_{ij} = \alpha_i(1, 5) - \alpha_j(1, 5)$ is a cycle, we see that $\int_{\alpha_i(1,5)} \omega \equiv \int_{\alpha_j(1,5)} \omega \pmod{\Omega\mathbb{Z}^6 + \mathbb{Z}^6}$. And we have

$$\int_{D_{12}+D_{15}} \varphi_1 = \int_{2\alpha_1(1,5)-\alpha_2(1,5)-\alpha_5(1,5)} \varphi_1 = (2 - \zeta^2 - \zeta^3) \int_{\alpha_1(1,5)} \varphi_1.$$

By the same calculation, we see that

$$\begin{aligned} \int_{\alpha_1(1,5)} \varphi_k &= \begin{cases} \frac{1}{5}(2 - \zeta - \zeta^4) \int_{D_{12}+D_{15}} \varphi_k & (k = 1, 2, 3) \\ \frac{1}{5}(2 - \zeta^2 - \zeta^3) \int_{D_{12}+D_{15}} \varphi_k & (k = 4, 5, 6) \end{cases} \\ &= \frac{1}{5} \int_{[2-\rho^2-\rho^3](D_{12}+D_{15})} \varphi_k. \end{aligned}$$

Calculating intersection numbers, we have the following equality

$$[2 - \rho^2 - \rho^3](D_{12} + D_{15}) = 2A_1 + 2A_3 + A_4 + A_6 + 4B_1 + 4B_3 - B_4 - B_6$$

as homology classes. Hence it holds

$$\begin{aligned} \int_{\alpha_1(1,5)} \omega &\equiv \frac{1}{5} \int_{2A_1+2A_3+A_4+A_6+4B_1+4B_3-B_4-B_6} \omega \\ &\equiv \frac{1}{10} \int_{-6A_1-6A_3-8A_4-8A_6+8B_1+8B_3+8B_4+8B_6} \omega \equiv (8, 0, 8). \end{aligned}$$

By the same way, we obtain the results for $\alpha_k(1, 2)$, $\alpha_k(1, 3)$ and $\alpha_k(1, 4)$. \square

Let C_λ ($\lambda \in X^\circ(2, 5)$) be any element of our family \mathcal{F} . We defined in Section 2 the system $\{\alpha_k(i, j)(\lambda)\}$, $\{\gamma_i(\lambda)\}$ and $\{A_i(\lambda), B_i(\lambda)\}$ on C_λ depending on the arc r . The point P_0 has always the same meaning. So Lemma 5.4 and 5.5 are true for C_λ using these notations. Let $\Delta \equiv (5, 5, 5)$ denote the Riemann constant on C_λ .

Now, recall that \mathbb{B}_2^0 , $\ell(ij)$ stands for $\Phi(X^\circ(2, 5))$ and $\pi^{-1}(\tilde{\Phi}(L(ij)))$ respectively (see Section 4).

Proof. Let us consider a curve $C = C_\lambda$ ($\lambda \in X^\circ(2, 5)$) and its period $\Omega = \Omega_\lambda$. We assume that $\Theta_{(1,1,1)}(\Omega) = 0$. According to Corollary 5.1, there exist points $Q_1, \dots, Q_5 \in C$ such that

Hence $\vartheta(a_1, a_2, a_3)$ is the union of certain $\ell(ij)$'s.

Lemma 5.6. *Let η_0 be the point $[0 : 0 : 1] \in \mathbb{B}_2^A$, and let (a_1, a_2, a_3) be a member of (5.10). If $a_1, a_2, a_3 \in \{1, 9\}$, then we have $\Theta_{(a_1, a_2, a_3)}(\Omega(\eta_0)) \neq 0$.*

Proof. Let

$$(a', b') = \left(\frac{1}{10} {}^t(\alpha, \alpha), \frac{1}{10} {}^t(-2\alpha, -\alpha) \right)$$

be a characteristic in $(\mathbb{Q}^2)^2$. Let $\Theta_\alpha(\tau)$ denote the theta constant $\Theta_{(a', b')}(\tau)$ ($\tau \in \mathfrak{S}_2$). Using this notation, we have

$$(5.14) \quad \Theta_{(a_1, a_2, a_3)}(\Omega(\eta_0)) = \Theta_{a_1}(\tau_0) \Theta_{a_2}(\tau_0) \Theta_{a_3}(\tau_0), \quad \tau_0 = \begin{pmatrix} \zeta - 1 & \zeta^2 + \zeta^3 \\ \zeta^2 + \zeta^3 & -\zeta^4 \end{pmatrix}$$

(see (4.4)). So our assertion is reduced to the inequality $\Theta_1(\tau_0) \neq 0$, since Θ_9 is a constant multiple of Θ_1 . Set

$$a = {}^t\left(\frac{1}{10}, \frac{1}{10}\right), \quad b = {}^t\left(-\frac{2}{10}, -\frac{1}{10}\right), \quad n = {}^t(n_1, n_2),$$

and set

$$f(n_1, n_2) = \exp[\pi\sqrt{-1}({}^t(n+a)\tau_0(n+a) + 2{}^t(n+a)b)].$$

By definition, $\Theta_1(\tau_0) = \sum_{n_1, n_2 \in \mathbb{Z}} f(n_1, n_2)$. For simplicity, we denote $n+a$ by $m = (m_1, m_2)$. By elementary calculations, we see that

$$|f(n_1, n_2)| = \exp[-\pi \sin\left(\frac{2\pi}{5}\right)\{m_1^2 + (3 - \sqrt{5})m_1m_2 + m_2^2\}].$$

In case $m_1m_2 > 0$, we have

$$|f(n_1, n_2)| < \exp[-\pi \sin\left(\frac{2\pi}{5}\right)\{m_1^2 + m_2^2\}].$$

In case $m_1m_2 < 0$, we have

$$\begin{aligned} |f(n_1, n_2)| &< \exp[-\pi \sin\left(\frac{2\pi}{5}\right)\{m_1^2 + m_1m_2 + m_2^2\}] \\ &= \exp[-\pi \sin\left(\frac{2\pi}{5}\right)\{\frac{1}{2}(m_1^2 + m_2^2) + \frac{1}{2}(m_1 + m_2)^2\}] \\ &< \exp[-\frac{\pi}{2} \sin\left(\frac{2\pi}{5}\right)\{m_1^2 + m_2^2\}]. \end{aligned}$$

Consequently,

$$|f(n_1, n_2)| < \alpha^{m_1^2 + m_2^2}, \quad (\alpha = \exp[-\frac{\pi}{2} \sin\left(\frac{2\pi}{5}\right)])$$

for any $n_1, n_2 \in \mathbb{Z}$. Set

$$D_1 = \{(n_1, n_2) \in \mathbb{Z}^2 : -10 \leq n_1, n_2 \leq 10\}, \quad D_2 = \mathbb{Z}^2 - D_1,$$

and consider the summations

$$S_1 = \sum_{D_1} f(n_1, n_2), \quad S_2 = \sum_{D_2} f(n_1, n_2).$$

Using a computer, we can evaluate $|S_1|$ and $|S_2|$. We have a approximate value

$$|S_1| \doteq 1.13746 \cdots,$$

by *Mathematica*. On the other hand, we have

$$|S_2| < \sum_{D_2} |f(n_1, n_2)| < \sum_{D_2} \alpha^{m_1^2 + m_2^2}.$$

The last term is very small. For example,

$$\sum_{n_1 \geq 10, n_2 \geq 0} \alpha^{m_1^2 + m_2^2} < (\sum_{n_1 \geq 10} \alpha^{n_1}) (\sum_{n_2 \geq 0} \alpha^{n_2}) = (\frac{\alpha^{10}}{1 - \alpha}) (\frac{\alpha}{1 - \alpha}) \doteq 5.40545 \times 10^{-7},$$

and the same calculations shows $|S_1| \gg |S_2|$. This implies $\Theta_1(\tau_0) = S_1 + S_2 \neq 0$. \square

Lemma 5.7. (1) If we have $a_1 \equiv 3, 7 \pmod{10}$, then $\Theta_{(a_1, a_2, a_3)}$ vanishes on $\ell(12)$.
(2) If we have $a_2 \equiv 3, 7 \pmod{10}$, then $\Theta_{(a_1, a_2, a_3)}$ vanishes on $\ell(34)$.

Proof. Set $g = \hat{g}_{12} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and set $\Omega = \Omega(\eta)$ with $\eta = [0 : \eta_2 : \eta_3] \in \mathbb{B}_2^A$. By the computation same as the one in the proof of Lemma 5.2, we have

$$g\Omega = \Omega, \quad \det(C\Omega + D) = \zeta, \quad \phi_{(a_1, a_2, a_3)}(g) = \frac{1}{40} a_1^2, \quad \Theta_{g(a_1, a_2, a_3)}(\Omega) = \Theta_{(a_1, a_2, a_3)}(\Omega).$$

Hence it holds

$$\Theta_{(a_1, a_2, a_3)}(\Omega)^8 = \exp\left[\frac{2}{5}\pi\sqrt{-1}(a_1^2 - 1)\right] \Theta_{(a_1, a_2, a_3)}(\Omega)^8$$

Therefore $\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))$ vanishes on the mirror of g_{12} provided $a_1 \equiv 3, 7 \pmod{10}$. This implies assertion (1). The assertion (2) follows by the same argument with $g = g_{34}$ and $\eta = [\eta_1 : 0 : \eta_3] \in \mathbb{B}_2^A$. \square

Proposition 5.2. We have the Table 1 for the vanishing loci of twelve theta constants coming from the system (5.10). In the table, “v” implies that $\Theta_{(a_1, a_2, a_3)}$ vanishes there, and the blank implies $\Theta_{(a_1, a_2, a_3)}$ is not identically zero there. For example, $\Theta_{(1, 1, 1)}$ vanishes on $\ell(13)$ and is not identically zero on $\ell(12)$.

(a_1, a_2, a_3)	$\ell(12)$	$\ell(13)$	$\ell(14)$	$\ell(15)$	$\ell(23)$	$\ell(24)$	$\ell(25)$	$\ell(34)$	$\ell(35)$	$\ell(45)$
(1, 1, 1)		v		v	v	v				v
(1, 1, 9)		v	v			v	v		v	
(1, 9, 1)			v	v	v	v			v	
(9, 1, 1)		v	v		v		v			v
(1, 3, 5)			v	v	v		v	v		
(1, 7, 5)		v		v		v	v	v		
(3, 1, 5)	v		v		v				v	v
(7, 1, 5)	v	v				v			v	v
(3, 3, 3)	v		v				v	v	v	
(3, 3, 7)	v			v	v			v		v
(3, 7, 3)	v	v					v	v		v
(7, 3, 3)	v			v		v		v	v	

TABLE 1.

Proof. By Lemma 5.7,

$$\Theta_{(3, 1, 5)}, \quad \Theta_{(7, 1, 5)}, \quad \Theta_{(3, 3, 3)}, \quad \Theta_{(3, 3, 7)}, \quad \Theta_{(3, 7, 3)}, \quad \Theta_{(7, 3, 3)}$$

vanish on $\ell(12)$, and

$$\Theta_{(1, 3, 5)}, \quad \Theta_{(1, 7, 5)}, \quad \Theta_{(3, 3, 3)}, \quad \Theta_{(3, 3, 7)}, \quad \Theta_{(3, 7, 3)}, \quad \Theta_{(7, 3, 3)}$$

vanish on $\ell(34)$. By Lemma 5.6,

$$\Theta_{(1,1,1)}, \quad \Theta_{(1,1,9)}, \quad \Theta_{(1,9,1)}, \quad \Theta_{(9,1,1)}$$

are not identically zero on $\ell(12)$ and on $\ell(34)$, since $\eta_0 = [0 : 0 : 1] \in \ell(12) \cap \ell(34)$. The result is obtained by applying the transformation formula (5.8) for above theta constants and \hat{g}_{ij} . For example, we have

$$\Theta_{\hat{g}_{12}\hat{g}_{45}(a_1,a_2,a_3)}(\hat{g}_{12}\hat{g}_{45}\Omega) = (\text{a unit function}) \times \Theta_{(a_1,a_2,a_3)}(\Omega).$$

Since $\hat{g}_{12}\hat{g}_{45}(1, 3, 5) \equiv (9, 9, 1)$ (see Lemma 5.3) and $g_{12}g_{45}(\ell(12)) = \ell(12)$, we see that $\Theta_{(1,3,5)}$ is not identically zero on $\ell(12)$. \square

5.3. Automorphic Factor. We study the automorphic factor appeared in the transformation formula (5.8) with respect to $\Gamma(1 - \zeta)$ and $\Omega = \Omega(\eta)$. Let H be the diagonal matrix $\text{diag}(1, 1, -\zeta^3(1 + \zeta))$. We denote ${}^t\eta H \eta$ by $\langle \eta, \eta \rangle$. Set

$$F_g(\eta) = \frac{\langle g\eta, g\eta \rangle}{\langle \eta, \eta \rangle}$$

for $g \in \Gamma$ and $\eta \in \mathbb{B}_2^A$. Obviously, we have the following lemma.

Lemma 5.8. *$F_g(\eta)$ satisfies the cocycle condition with respect to Γ . That is,*

$$F_{g_1 g_2}(\eta) = F_{g_1}(g_2 \eta) F_{g_2}(\eta), \quad g_1, g_2 \in \Gamma.$$

Proposition 5.3. *There exist the non trivial character*

$$\chi : \Gamma \longrightarrow \mu_5 = \{1, \zeta, \dots, \zeta^4\}$$

such that

$$\det(C\Omega(\eta) + D) = \chi(g) F_g(\eta) \quad (\eta \in \mathbb{B}_2^A)$$

for $g \in \Gamma$, where the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the symplectic representation \hat{g} of g .

Proof. According to the case by case calculation, we have

$$\det(C\Omega(\eta) + D) = \zeta^3 F_g(\eta) \quad (\eta \in \mathbb{B}_2^A)$$

for $g = g_{12}, g_{23}, g_{34}, g_{45}$. Since $\det(C\Omega(\eta) + D)/F_g(\eta)$ satisfies the cocycle condition, we obtain the result. \square

Now let (a, b) be a invariant characteristic (a_1, a_2, a_3) , and (a_g, b_g) be $\hat{g}(a, b)$ for $g \in \Gamma(1 - \zeta)$. Since

$$(a_g, b_g) \equiv (a, b) \pmod{\mathbb{Z}},$$

we have

$$\Theta_{\hat{g}(a,b)}(\Omega) = \Theta_{(a_g,b_g)}(\Omega) = \Theta_{(a_g-a+a, b_g-b+b)}(\Omega) = \exp[2\pi\sqrt{-1}{}^t a(b_g - b)] \Theta_{(a,b)}(\Omega)$$

by (5.3). Set

$$\phi'_{(a_1,a_2,a_3)}(\hat{g}) = \phi_{(a_1,a_2,a_3)}(\hat{g}) - {}^t a(b_g - b).$$

Then we can write the transformation formula (5.8) as

$$(5.15) \quad \Theta_{(a_1,a_2,a_3)}(\Omega(g\eta)) = \kappa(\hat{g}) \exp(2\pi\sqrt{-1}\phi'_{(a_1,a_2,a_3)}(\hat{g})) [\chi(g) F_g(\eta)]^{\frac{1}{2}} \Theta_{(a_1,a_2,a_3)}(\Omega(\eta)),$$

where $\kappa(\hat{g})$ is a 8-th root of 1 depending only on \hat{g} .

Lemma 5.9. *Let g be in $\Gamma(1 - \zeta)$. Then, the values*

$$[\exp(2\pi\sqrt{-1}\phi'_{(a_1, a_2, a_3)}(\hat{g}))]^5$$

are the same for all twelve characteristics (a_1, a_2, a_3) in (5.10).

Proof. By direct calculations, we have

$$\begin{aligned} 5\phi'_{(a_1, a_2, a_3)}(\hat{h}_{12}) &\equiv \frac{1}{8}, & 5\phi'_{(a_1, a_2, a_3)}(\hat{h}_{13}) &\equiv \frac{3}{4}, & 5\phi'_{(a_1, a_2, a_3)}(\hat{h}_{14}) &\equiv \frac{1}{2}, \\ 5\phi'_{(a_1, a_2, a_3)}(\hat{h}_{23}) &\equiv \frac{1}{2}, & 5\phi'_{(a_1, a_2, a_3)}(\hat{h}_{34}) &\equiv \frac{3}{4} \pmod{\mathbb{Z}} \end{aligned}$$

for the twelve (a_1, a_2, a_3) . According to Lemma 5.8, the equality (5.15) shows that

$$\kappa(\hat{g}) \exp[2\pi\sqrt{-1}\phi'_{(a_1, a_2, a_3)}(\hat{g})]$$

is a character on $\Gamma(1 - \zeta)$. So we obtain the result for any $g \in \Gamma(1 - \zeta)$. \square

Corollary 5.3. *Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be in (5.10). Then, the function*

$$\frac{\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5}{\Theta_{(b_1, b_2, b_3)}(\Omega(\eta))^5}$$

is well-defined as meromorphic function on $\mathbb{B}_2^A/\Gamma(1 - \zeta)$.

Let $\Omega = \Omega_\lambda$ be the period matrix of a curve C_λ ($\lambda \in X^\circ(2, 5)$), P_0 be a ramified point of $C \rightarrow \mathbb{P}^1$.

Proposition 5.4. *Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be in (5.10). The function*

$$f(P) = \frac{\Theta_{(a_1, a_2, a_3)}(\int_{P_0}^P \omega, \Omega)^5}{\Theta_{(b_1, b_2, b_3)}(\int_{P_0}^P \omega, \Omega)^5} \quad (P \in C_\lambda)$$

is a single-valued meromorphic function on C_λ , where the paths of integrations in the numerator and the denominator are chosen as same.

Proof. Note that Corollary 5.2 asserts

$$\Theta_{(a_1, a_2, a_3)}(\int_{P_0}^{P_0} \omega, \Omega) = \text{const.} \times \Theta_{(a_1, a_2, a_3)}(0, \Omega) \neq 0,$$

where the constant depends on the path of integration. So the numerator is not identically zero, and it is same for the denominator. By the assumption we have

$$(a_1, a_2, a_3) - (b_1, b_2, b_3) \in (\frac{1}{5}\mathbb{Z}^6)^2.$$

By using the formula (5.1) and (5.2) we can check that

$$\frac{\Theta_{(a_1, a_2, a_3)}(\int_{P_0}^P \omega + \Omega m + n, \Omega)^5}{\Theta_{(b_1, b_2, b_3)}(\int_{P_0}^P \omega + \Omega m + n, \Omega)^5} = \frac{\Theta_{(a_1, a_2, a_3)}(\int_{P_0}^P \omega, \Omega)^5}{\Theta_{(b_1, b_2, b_3)}(\int_{P_0}^P \omega, \Omega)^5}$$

for $m, n \in \mathbb{Z}^6$. This implies single-valuedness of f . \square

Let us consider the meromorphic function

$$f(P) = \frac{\Theta_{(1,1,1)}(\int_{P_1}^P \omega, \Omega)^5}{\Theta_{(3,3,7)}(\int_{P_1}^P \omega, \Omega)^5}$$

on C_λ . By Lemma 5.5, we have

$$\begin{aligned} \Delta - (1, 1, 1) &\equiv (4, 4, 4) \equiv 2 \int_{P_1}^{P_2} \omega + 3 \int_{P_1}^{P_3} \omega + \int_{P_1}^{P_4} \omega, \\ \Delta - (3, 3, 7) &\equiv (2, 2, 8) \equiv 3 \int_{P_1}^{P_2} \omega + 2 \int_{P_1}^{P_3} \omega + \int_{P_1}^{P_4} \omega. \end{aligned}$$

By Corollary 5.1, the zero divisor of $\Theta_{(1,1,1)}(\int_{P_1}^P \omega, \Omega)$ and $\Theta_{(3,3,7)}(\int_{P_1}^P \omega, \Omega)$ are $2P_2 + 3P_3 + P_4$ and $3P_2 + 2P_3 + P_4$ respectively. Hence we can write

$$f(P) = c \frac{x(P) - \lambda_3}{x(P) - \lambda_2},$$

where $x(P)$ is the coordinate function $x \in \mathbb{C}[x, y]/(y^5 - \prod(x - \lambda_i))$ and $c \neq 0$ is a certain constant. By Lemma 5.5,

$$\int_{P_1}^{P_1} \omega \equiv (0, 0, 0), \quad \int_{P_1}^{P_5} \omega \equiv (8, 0, 8).$$

Substitutes $P = P_1, P_5$ in the above form, then we obtain

$$\frac{\Theta_{(1,1,1)}((0, 0, 0), \Omega)^5}{\Theta_{(3,3,7)}((0, 0, 0), \Omega)^5} = c \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}, \quad \frac{\Theta_{(1,1,1)}((8, 0, 8), \Omega)^5}{\Theta_{(3,3,7)}((8, 0, 8), \Omega)^5} = c \frac{\lambda_5 - \lambda_3}{\lambda_5 - \lambda_2}.$$

Set $(8, 0, 8) = \Omega\varepsilon' + \varepsilon''$. By elementary and patient calculation, we have

$$\begin{aligned} \Theta_{(1,1,1)}((8, 0, 8), \Omega)^5 &= -\zeta^2 \exp[-5\pi\sqrt{-1}^t \varepsilon' \Omega \varepsilon' - 10\pi\sqrt{-1}^t \varepsilon' \varepsilon''] \Theta_{(1,9,1)}(\Omega)^5 \\ \Theta_{(3,3,7)}((8, 0, 8), \Omega)^5 &= \exp[-5\pi\sqrt{-1}^t \varepsilon' \Omega \varepsilon' - 10\pi\sqrt{-1}^t \varepsilon' \varepsilon''] \Theta_{(1,3,5)}(\Omega)^5 \end{aligned}$$

Eliminating c , we have the following equality

$$\frac{\Theta_{(1,1,1)}(\Omega)^5 \Theta_{(1,3,5)}(\Omega)^5}{\Theta_{(3,3,7)}(\Omega)^5 \Theta_{(1,9,1)}(\Omega)^5} = -\zeta^2 \frac{(\lambda_1 - \lambda_3)(\lambda_5 - \lambda_2)}{(\lambda_1 - \lambda_2)(\lambda_5 - \lambda_3)}.$$

Note that we can regard the above equality as that of meromorphic functions on $\mathbb{B}_2^A/\Gamma(1 - \zeta) \cong X(2, 5)$. By the above equality and Proposition 5.2, we see that

1. The vanishing order of $\Theta_{(1,1,1)}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(13))$ is 1,
2. The vanishing order of $\Theta_{(1,3,5)}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(25))$ is 1,
3. The vanishing order of $\Theta_{(3,3,7)}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(12))$ is 1,
4. The vanishing order of $\Theta_{(1,9,1)}(\Omega(\eta))^5$ on $\tilde{\Phi}(L(35))$ is 1.

Because Γ acts transitively on the set of σ -invariant characteristics (see Lemma 5.3), we obtain the following result.

Proposition 5.5. *Let (a_1, a_2, a_3) be a σ -invariant characteristic. If the multi-valued function $\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5$ on $\mathbb{B}_2^A/\Gamma(1 - \zeta)$ identically vanishes on $\tilde{\Phi}(L(ij)) = \ell(ij)/\Gamma(1 - \zeta)$, then the vanishing order is 1.*

6. CONCLUSION

Now we state our results.

— **The Schwarz inverse for the Appell HGDE** $F_1(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{6}{5})$ —

Recall the embedding of $J : X(2, 5) \rightarrow \mathbb{P}^{11}$ with

$$J(ijklm)(\lambda) = (\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_l)(\lambda_l - \lambda_m)(\lambda_m - \lambda_i)$$

in Proposition 1.1, and the extended period map $\tilde{\Phi}$ in section 4.

Theorem 6.1. *We have a commutative diagram:*

$$\begin{array}{ccc} X(2, 5) & \xrightarrow{\tilde{\Phi}} & \mathbb{B}_2^A / \Gamma(1 - \zeta) \\ \downarrow J & \searrow \Theta & \\ \mathbb{P}^{11} & & \end{array}$$

FIGURE 4.

by putting

$$(6.1) \quad \Theta = \begin{bmatrix} \Theta_{(1,1,1)}(\Omega(\eta))^5 \\ \Theta_{(1,1,9)}(\Omega(\eta))^5 \\ \Theta_{(1,9,1)}(\Omega(\eta))^5 \\ \Theta_{(9,1,1)}(\Omega(\eta))^5 \\ \Theta_{(1,3,5)}(\Omega(\eta))^5 \\ \Theta_{(1,7,5)}(\Omega(\eta))^5 \\ \Theta_{(3,3,3)}(\Omega(\eta))^5 \\ \Theta_{(3,3,7)}(\Omega(\eta))^5 \\ \Theta_{(3,7,3)}(\Omega(\eta))^5 \\ \Theta_{(7,3,3)}(\Omega(\eta))^5 \\ \Theta_{(7,1,5)}(\Omega(\eta))^5 \\ \Theta_{(3,1,5)}(\Omega(\eta))^5 \end{bmatrix}, \quad J = \begin{bmatrix} c_1 J(13245)(\lambda) \\ c_2 J(13524)(\lambda) \\ c_3 J(15324)(\lambda) \\ c_4 J(13254)(\lambda) \\ c_5 J(15234)(\lambda) \\ c_6 J(13425)(\lambda) \\ d_1 J(12534)(\lambda) \\ d_2 J(12345)(\lambda) \\ d_3 J(13452)(\lambda) \\ d_4 J(15342)(\lambda) \\ d_5 J(12453)(\lambda) \\ d_6 J(12354)(\lambda) \end{bmatrix},$$

with constants

$$[c_1 : \cdots : c_6 : d_1 : \cdots : d_6] = [1 : -1 : 1 : 1 : \zeta^3 : \zeta^3 : -\zeta : \zeta : \zeta : -\zeta : -1 : -1] \in \mathbb{P}^{11}.$$

Moreover the map Θ is an embedding.

Proof. By Proposition 5.2 and Proposition 5.5, the zero divisor of the i -th component of Θ coincides with that of the i -th component of J via the isomorphism $\tilde{\Phi}$. So we can write as

$$(6.2) \quad \frac{\Theta_{(1,1,1)}(\Omega(\tilde{\Phi}(\lambda)))^5}{J(13245)(\lambda)} = c_1, \dots, \frac{\Theta_{(3,1,5)}(\Omega(\tilde{\Phi}(\lambda)))^5}{J(12354)(\lambda)} = d_6$$

with certain constants c_1, \dots, d_6 . It shows the diagram in question is commutative. Since J is an embedding and $\tilde{\Phi}$ is an isomorphism, we see that Θ is an embedding.

Next we determine the ratios of the constants c_i, d_i .

Let $\eta \in \mathbb{B}_2^A$ be a point of the form $[0 : \eta_2 : \eta_3]$. For such η , we have the decomposition

$$\Omega(\eta) = \tau_0 \oplus \Omega(\eta)'$$

(see (4.3)). So we have the splitting

$$\Theta_{(a_1, a_2, a_3)}(\Omega(\eta)) = \Theta_{a_1}(\tau_0) \Psi_{(a_2, a_3)}(\Omega(\eta)'),$$

where $\Theta_{a_1}(\tau_0)$ is same as in the proof of Lemma 5.6, and $\Psi_{(a_2, a_3)}(\Omega(\eta)'),$ is the theta function $\Theta_{(a, b)}(\Omega(\eta)'),$ of degree 4 with the characteristic

$$a = \frac{1}{10}(a_2, a_3, a_2, a_3), \quad b = \frac{1}{10}(-2a_2, -2a_3, -a_2, -a_3).$$

By the same way, we have

$$\Theta_{(a_1, a_2, a_3)}(\Omega(\eta)) = \Theta_{a_2}(\tau_0) \Psi_{(a_1, a_3)}(\Omega(\eta)'')$$

for $\eta = [\eta_1 : 0 : \eta_3] \in \mathbb{B}_2^A$. We can check that

$$\Theta_1(\tau_0)^5 = -\Theta_9(\tau_0)^5, \quad \Psi_{(1,9)}(\Omega(\eta)')^5 = \Psi_{(9,1)}(\Omega(\eta)')^5, \quad \Psi_{(3,5)}(\Omega(\eta)')^5 = \Psi_{(7,5)}(\Omega(\eta)')^5$$

by (5.3) and (5.4). So we have

$$(6.3) \quad \frac{\Theta_{(1,1,1)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5} = \frac{\Theta_1(\tau_0)^5 \Psi_{(1,1)}(\Omega(\eta)')^5}{\Theta_9(\tau_0)^5 \Psi_{(1,1)}(\Omega(\eta)')^5} = -1 \quad \text{on } \ell(12).$$

By the same way, we see

$$(6.4) \quad \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,9,1)}(\Omega(\eta))^5} = 1, \quad \frac{\Theta_{(1,3,5)}(\Omega(\eta))^5}{\Theta_{(1,7,5)}(\Omega(\eta))^5} = 1 \quad \text{on } \ell(12),$$

and

$$(6.5) \quad \frac{\Theta_{(1,1,1)}(\Omega(\eta))^5}{\Theta_{(1,9,1)}(\Omega(\eta))^5} = -1, \quad \frac{\Theta_{(9,1,1)}(\Omega(\eta))^5}{\Theta_{(1,1,9)}(\Omega(\eta))^5} = 1, \quad \frac{\Theta_{(3,1,5)}(\Omega(\eta))^5}{\Theta_{(7,1,5)}(\Omega(\eta))^5} = 1 \quad \text{on } \ell(34).$$

Moreover, we have

$$(6.6) \quad \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} = \frac{\Theta_1(\tau_0)^5 \Theta_1(\tau_0)^5 \Theta_9(\tau_0)^5}{\Theta_1(\tau_0)^5 \Theta_1(\tau_0)^5 \Theta_1(\tau_0)^5} = -1 \quad \text{for } \eta \in \ell(12) \cap \ell(34)$$

(see (5.14)). By the transformation formula (5.8), we have

$$\frac{\Theta_{\hat{g}(a_1, a_2, a_3)}(\Omega(g\eta))^5}{\Theta_{\hat{g}(b_1, b_2, b_3)}(\Omega(g\eta))^5} = \exp[2\pi\sqrt{-1}\{\phi_{(a_1, a_2, a_3)}(\hat{g}) - \phi_{(b_1, b_2, b_3)}(\hat{g})\}] \frac{\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5}{\Theta_{(b_1, b_2, b_3)}(\Omega(\eta))^5}$$

for any pair of σ -invariant characteristics $(a_1, a_2, a_3), (b_1, b_2, b_3)$, and $g \in \Gamma$. By explicit calculation of the above formula, we obtain

$$\begin{aligned} \frac{\Theta_{(1,1,1)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5} &= -\zeta^4 \frac{\Theta_{(3,3,7)}(\Omega(g_{23}\eta))^5}{\Theta_{(3,1,5)}(\Omega(g_{23}\eta))^5}, & \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,9,1)}(\Omega(\eta))^5} &= \zeta^2 \frac{\Theta_{(3,3,3)}(\Omega(g_{23}\eta))^5}{\Theta_{(1,3,5)}(\Omega(g_{23}\eta))^5}, \\ \frac{\Theta_{(1,3,5)}(\Omega(\eta))^5}{\Theta_{(1,7,5)}(\Omega(\eta))^5} &= \zeta \frac{\Theta_{(1,9,1)}(\Omega(g_{23}\eta))^5}{\Theta_{(7,3,3)}(\Omega(g_{23}\eta))^5}, & \frac{\Theta_{(3,1,5)}(\Omega(\eta))^5}{\Theta_{(7,1,5)}(\Omega(\eta))^5} &= \zeta \frac{\Theta_{(9,1,1)}(\Omega(g_{23}\eta))^5}{\Theta_{(3,7,3)}(\Omega(g_{23}\eta))^5}, \\ \frac{\Theta_{(3,1,5)}(\Omega(\eta))^5}{\Theta_{(7,1,5)}(\Omega(\eta))^5} &= -\frac{\Theta_{(3,3,7)}(\Omega(g_{45}\eta))^5}{\Theta_{(3,7,3)}(\Omega(g_{45}\eta))^5}, & \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} &= \zeta^2 \frac{\Theta_{(1,7,5)}(\Omega(g_{45}\eta))^5}{\Theta_{(9,1,1)}(\Omega(g_{45}\eta))^5}. \end{aligned}$$

Comparing these with (6.3), (6.4), (6.5) and (6.6), we have

$$(6.7) \quad \frac{\Theta_{(3,3,7)}(\Omega(\eta))^5}{\Theta_{(3,1,5)}(\Omega(\eta))^5} = \zeta, \quad \frac{\Theta_{(3,3,3)}(\Omega(\eta))^5}{\Theta_{(1,3,5)}(\Omega(\eta))^5} = \zeta^3, \quad \frac{\Theta_{(1,9,1)}(\Omega(\eta))^5}{\Theta_{(7,3,3)}(\Omega(\eta))^5} = \zeta^4 \quad \text{on } \ell(13),$$

$$(6.8) \quad \frac{\Theta_{(9,1,1)}(\Omega(\eta))^5}{\Theta_{(3,7,3)}(\Omega(\eta))^5} = \zeta^4 \quad \text{on } \ell(24),$$

$$(6.9) \quad \frac{\Theta_{(3,3,7)}(\Omega(\eta))^5}{\Theta_{(3,7,3)}(\Omega(\eta))^5} = -1, \quad \text{on } \ell(35),$$

$$(6.10) \quad \frac{\Theta_{(1,7,5)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5} = \zeta^3 \quad \text{on } \ell(12) \cap \ell(35),$$

since it holds

$$g_{23}(\ell(12)) = \ell(13), \quad g_{23}(\ell(34)) = \ell(24), \quad g_{45}(\ell(12)) = \ell(12), \quad g_{45}(\ell(34)) = \ell(35).$$

Because the commutativity of diagram is established, by using (6.3), we see that

$$\begin{aligned} -1 &= \frac{\Theta_{(1,1,1)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5} \Big|_{\ell(12)} = \frac{c_1}{c_4} \frac{J(13245)}{J(13254)} \Big|_{L(12)} \\ &= \frac{c_1}{c_4} \frac{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_4)(\lambda_4 - \lambda_5)(\lambda_5 - \lambda_1)}{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_5)(\lambda_5 - \lambda_4)(\lambda_4 - \lambda_1)}, \end{aligned}$$

that is $c_1 = c_4$. By the same calculation using (6.4)–(6.10), we obtain

$$\begin{aligned} c_1 &= c_4, \quad c_2 = -c_3, \quad c_5 = c_6, \quad c_1 = c_3, \quad c_2 = -c_4, \quad d_5 = d_6, \quad c_1 = -c_2, \\ d_2 &= -\zeta d_6, \quad d_1 = -\zeta^3 c_5, \quad c_3 = -\zeta^4 d_4, \quad c_4 = \zeta^4 d_3, \quad d_2 = d_3, \quad c_6 = \zeta^3 c_4. \end{aligned}$$

These equalities gives the ratios in the assertion. \square

Remark 6.1. *We have the following equalities;*

$$J(ijklm) = -J(mlkji), \quad \Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5 = -\Theta_{(10-a_1, 10-a_2, 10-a_3)}(\Omega(\eta))^5.$$

Let us denote $X(2, 5)$ by X , and let K_X be the canonical class of X .

Corollary 6.1. *We have an isomorphism of \mathbb{C} -algebras*

$$\mathbb{C}[\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5] \cong \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(-nK_X)),$$

where the left hand side is the \mathbb{C} -algebra of the functions on \mathbb{B}_2^A generated by the twelve theta functions in Theorem 6.1. Especially the \mathbb{C} -vector space spanned by $\{\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5\}$ coincides with $H^0(X, \mathcal{O}_X(-K_X))$.

Proof. The map J is essentially anti-canonical map (see Section 1). Hence it follows from Theorem 6.1. \square

Remark 6.2. *By the Riemann-Roch theorem, we obtain*

$$\begin{aligned} \dim H^0(X, \mathcal{O}_X(-nK_X)) &= \frac{1}{2}(-nK_X) \cdot (-nK_X - K_X) + 1 \\ &= \frac{5}{2}n(n+1) + 1 \end{aligned}$$

since $(-K_X) \cdot (-K_X) = 5$. So we have $\dim H^0(X, \mathcal{O}_X(-K_X)) = 6$, and twelve $\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5$ satisfy 6 linear equations. It is known that the image of X in \mathbb{P}^5 by the anti-canonical map is determined by the system of quadratic equations (see [2, Chapter 5]).

— **The graded ring of Automorphic forms** —

Recall the automorphic factor $F_g(\eta)$ in Lemma 5.8. We consider the automorphic function $f(\eta)$ on \mathbb{B}_2^A in the sense that we have

$$(6.11) \quad f(g\eta) = F_g(\eta)^k f(\eta) \quad \text{for } g \in \Gamma(1 - \zeta),$$

where k is a non negative integer. Let us denote the vector space of holomorphic functions satisfying (6.11) by $A_k(F_g)$.

Proposition 6.1. *Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be the member of the system in (5.10), then it holds*

$$\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5 \Theta_{(b_1, b_2, b_3)}(\Omega(\eta))^5 \in A_5(F_g).$$

Proof. By (5.15) and Lemma 5.9, we have

$$(6.12) \quad \begin{aligned} & \Theta_{(a_1, a_2, a_3)}(\Omega(g\eta))^5 \Theta_{(b_1, b_2, b_3)}(\Omega(g\eta))^5 \\ &= \kappa(g)^{10} \exp(2\pi\sqrt{-1}\phi'_{(a_1, a_2, a_3)}(g))^{10} F_g(\eta)^5 \Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5 \Theta_{(b_1, b_2, b_3)}(\Omega(g\eta))^5 \end{aligned}$$

for $g \in \Gamma(1 - \zeta)$. We must show

$$(6.13) \quad [\kappa(g) \exp(2\pi\sqrt{-1}\phi'_{(a_1, a_2, a_3)}(g))]^{10} = 1$$

for $g \in \Gamma(1 - \zeta)$. Let η_0 be the point

$$(\text{the mirror of } h_{12}) \cap (\text{the mirror of } h_{34}) = [0 : 0 : 1] \in \mathbb{B}_2^A.$$

Then η_0 is fixed by h_{12} and h_{34} . Moreover, $F_g(\eta) = \zeta$ for h_{12} and h_{34} . So we have

$$\Theta_{(1,1,1)}(\Omega(\eta_0))^{10} = [\kappa(g) \exp(2\pi\sqrt{-1}\phi'_{(1,1,1)}(g))]^{10} \Theta_{(1,1,1)}(\Omega(\eta_0))^{10} \quad (g = h_{12}, h_{34})$$

by (6.12). Since $\Theta_{(1,1,1)}(\Omega(\eta_0)) \neq 0$ (see Proposition 5.6), we obtain (6.13) for h_{12} and h_{34} . By the same way, we see that (6.13) holds for any member h_{ij} of the generator system of $\Gamma(1 - \zeta)$. Hence it holds for all $g \in \Gamma(1 - \zeta)$. \square

Theorem 6.2. (1). *We have the isomorphism of the \mathbb{C} -algebras:*

$$\begin{aligned} \oplus_{n=0}^{\infty} A_{5n}(F_g) &= \mathbb{C}[\Theta_{(a_1, a_2, a_3)}(\Omega(\eta))^5 \Theta_{(b_1, b_2, b_3)}(\Omega(\eta))^5] \\ &\cong \oplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(-2nK_X)). \end{aligned}$$

(2). $A_n(F_g) = \{0\}$ for $n \in \mathbb{N}$, $n \equiv 1, 2, 3, 4 \pmod{5}$.

Proof. By Proposition 6.1, a function $f \in A_5(F_g)$ defines the meromorphic function

$$\frac{f(\eta)}{\Theta_{(1,1,1)}(\Omega(\eta))^{10}}$$

on $\mathbb{B}_2^A/\Gamma(1 - \zeta)$. So, by Theorem 6.1, we have the isomorphism of \mathbb{C} -vector space:

$$A_{5n}(F_g) \cong H^0(X, \mathcal{O}_X(-2nK_X)) \quad \text{for } n \in \mathbb{N}.$$

Hence we have the assertion (1).

Next let us recall that X is the blow up of \mathbb{P}^2 at 4 points. We denote this blow up by $\pi : X \rightarrow \mathbb{P}^2$. Then the Neron-Severi group $\text{NS}(X)$ has the free generator E_1, E_2, E_3, E_4 and π^*H , where $\{E_i\}$ are the exceptional curves with respect to π , and H is a general line on \mathbb{P}^2 . For $n \notin 5\mathbb{Z}$, there is no divisor D on X such that $5D = -2nK_X$ since $-K_X = 3\pi^*H - E_1 - E_2 - E_3 - E_4$. This implies the assertion (2) since

$$A_n(F_g)^5 \subset A_{5n}(F_g) \cong H^0(X, \mathcal{O}_X(-2nK_X)).$$

□

— **The Schwarz inverse for the Gauss HGDE** $E_{2,1}(\frac{1}{5}, \frac{2}{5}, \frac{4}{5})$ —

Let us consider the 1-dimensional disk

$$\mathbb{B}_1 = \{\eta \in \mathbb{B}_2^A : \eta_1 = 0\},$$

and the degenerate period map

$$\begin{aligned} \Phi_{12} : L(12) \cong \mathbb{P}^1 &\longrightarrow \mathbb{B}_1, \quad t \mapsto [0 : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega], \\ \omega &= x^{-\frac{4}{5}}(x-1)^{-\frac{2}{5}}(x-t)^{-\frac{2}{5}}dx, \end{aligned}$$

as in Section 4 (the parameter λ is specialized as $(\lambda_1, \dots, \lambda_5) = (0, 0, 1, t, \infty)$). Set

$$\Gamma(1-\zeta)_1 = \{g \in \Gamma(1-\zeta) : g(\mathbb{B}_1) = \mathbb{B}_1\}.$$

As we mentioned in Section 4, this is the triangle group $\Delta(5, 5, 5)$ up to the center. Recall those are the Schwarz map and the monodromy group for Gauss hypergeometric differential equation $E_{2,1}(\frac{1}{5}, \frac{2}{5}, \frac{4}{5})$ (see (4.2)). We have the explicit discription of the inverse:

Theorem 6.3. *The map*

$$\Theta_{12} : \mathbb{B}_1/\Gamma(1-\zeta)_1 \longrightarrow \mathbb{P}^1, \quad \eta \mapsto [\Theta_{(1,1,1)}(\Omega(\eta))^5 : -\Theta_{(1,1,9)}(\Omega(\eta))^5]$$

is an isomorphism, and this is the inverse map of the Schwarz map

$$\Phi_{12} : \mathbb{P}^1 \longrightarrow \mathbb{B}_1/\Gamma(1-\zeta)_1, \quad [1 : t] \mapsto [0 : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega].$$

Proof. By Theorem 6.1, the restriction of meromorphic function

$$\frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5}$$

on $L(12)$ is of order 1. In fact, $L(12) \cap L(13) = L(12) \cap L(14) = L(12) \cap L(15) = \phi$, so the numerator vanishes at only $L(12) \cap L(35)$ with order 1, the denominator vanishes at only $L(12) \cap L(45)$ with order 1, and $L(12) \cap L(35) \neq L(12) \cap L(45)$ (see Section 1). Hence the map Θ_{12} is an isomorphism. Moreover, by Theorem 6.1, we have the equality

$$\frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} = -\frac{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_5)(\lambda_5 - \lambda_2)(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_1)}{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_5)(\lambda_5 - \lambda_1)},$$

on $\mathbb{B}_2^A/\Gamma(1-\zeta) \cong X(2, 5)$, and this induces the equality

$$\frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} = -\frac{(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_1)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_5)}$$

on $L(12)$. Putting $(\lambda_1, \lambda_3, \lambda_4, \lambda_5) = (0, 1, t, \infty)$, we obtain

$$\frac{(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_1)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_5)} = t.$$

□

Let us consider a holomorphic function f on \mathbb{B}_1 satisfying the condition:

$$f(g\eta) = F_g(\eta)^k f(\eta) \quad \text{for } g \in \Gamma(1-\zeta)_1,$$

and we denote the \mathbb{C} -vector space of such functions by $M_k(F_g)$.

Corollary 6.2. (1). *We have an isomorphism of \mathbb{C} -algebras:*

$$\begin{aligned} & \oplus_{n=0}^{\infty} M_{5n}(F_g) \\ &= \mathbb{C}[\Theta_{(1,1,1)}(\Omega(\eta))^{10}, \Theta_{(1,1,1)}(\Omega(\eta))^5 \Theta_{(1,1,9)}(\Omega(\eta))^5, \Theta_{(1,1,9)}(\Omega(\eta))^{10}] \\ &\cong \mathbb{C}[x_0^2, x_0 x_1, x_1^2] \\ &\cong \oplus_{n=0}^{\infty} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-nK_{\mathbb{P}^1})), \end{aligned}$$

where $[x_0 : x_1]$ is homogeneous coordinates of \mathbb{P}^1 .

(2). $M_n(F_g) = \{0\}$ for $n \in \mathbb{N}$, $n \equiv 1, 2, 3, 4 \pmod{5}$.

Proof. The assertion (1) is a direct consequence of Corollary 6.2 and Theorem 6.3. The assertion (2) is obtained by the same argument as the proof of Theorem 6.2. \square

Acknowledgments. I express my sincere thanks to Professor Hironori Shiga for advices during the preparation of this paper.

REFERENCES

- [1] P. Deligne and G. D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, Publ. Math I.H.E.S. 63(1986), 5-88.
- [2] R. Friedmann, Algebraic surfaces and holomorphic vector bundles, Springer(1998).
- [3] R. P. Holzapfel, Ball and surface arithmetics. Aspects of Mathematics, E29. Friedr. Vieweg & Sohn, Braunschweig(1998).
- [4] J. Igusa, Theta functions, Springer, Heidelberg, New-York(1972).
- [5] K. Matsumoto, On Modular Functions in 2 Variables Attached to a Family of Hyperelliptic Curves of Genus 3, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 16 (1989), no. 4, 557-578.
- [6] K. Matsumoto, Theta functions on the bounded symmetric domain of type $I_{2,2}$ and the period map of a 4-parameter family of K3 surfaces, Math. Ann. 295(1993), 383-409.
- [7] K. Matsumoto, Theta constants associated with the cyclic triple coverings of the complex projective line branching at six points, math.AG/0008025.
- [8] D. Mumford, Tata Lectures on Theta I, Birkhäuser, Boston-Basel-Stuttgart(1983).
- [9] H. Shiga, On the representation of the Picard modular function by θ constants I-II, Pub. R.I.M.S. Kyoto Univ. 24(1988), 311-360.
- [10] H. Shiga, One attempt to the K3 modular function I-II, Ann. Scuola Norm. Pisa, Ser. IV-Vol. VI(1979), 609-635, Ser. IV-Vol. VIII(1981), 157-182.
- [11] G. Shimura, On purely transcendental fields of automorphic functions of several complex variables. Osaka J. Math. 1(1964), 1-14.
- [12] K. Takeuchi, Arithmetic triangle group, J. Math. Soc. Japan, Vol. 29-No. 1(1977), 91-106.
- [13] T. Terada, Fonctions hypergéométriques F_1 et fonctions automorphes I-II, J. Math. Soc. Japan 35(1983), 451-475, 37(1985), 173-185.
- [14] J. Wolfart, Graduierte algebren automorpher formen zu dreiecksgruppen, Analysis 1, no. 3(1981), 177-190.
- [15] T. Yamazaki and M. Yashida, On Hirzebruch's Examples of Surfaces with $c_1^2 = 3c_2$, Math. Ann. 266(1984), 421-431.
- [16] M. Yoshida, Fuchsian differential equations, Aspects of Mathematics, E11. Friedr. Vieweg & Sohn, Braunschweig(1987).
- [17] M. Yoshida, Hypergeometric functions, my love, Aspects of Mathematics, E32. Friedr. Vieweg & Sohn, Braunschweig(1997).

DEPARTMENT OF MATHEMATICS & INFORMATICS, FACULTY OF SCIENCE, CHIBA UNIVERSITY,
1-33 YAYOI-CHO, INAGE-KU, CHIBA 263-8522, JAPAN

E-mail address: mkoike@math.s.chiba-u.ac.jp